Comparing return distributions of equity linked retirement provision plans with different capital guarantee mechanisms and fee structures

Uwe Wystup, MathFinance AG
joint work with Nils Detering and Andreas Weber

Lorentz Workshop
Quantitative Methods in Financial and Insurance Mathematics
Leiden, 20 April 2011

Outline

- In Germany, a retirement plan, called Riester-Rente, is supported by the state with cash payments and tax benefits.
- Those retirement plans have to preserve the invested capital.
- Since investors want to be invested in the equity market there is an increasing demand for guarantee concepts for long term equity investments.

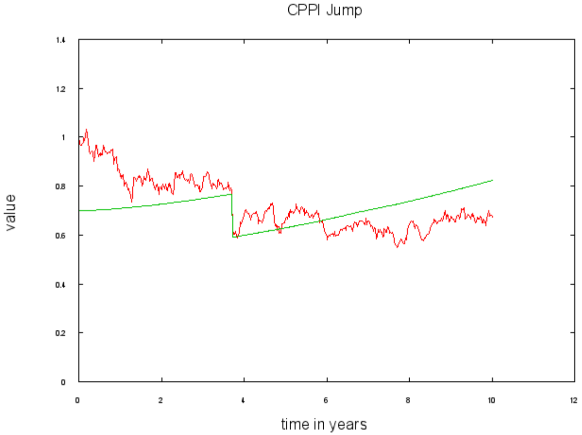
Outline

- Which different return distributions are generated by the different strategies in a reasonable model?
- How big is the risk of failing to generate the guarantee if we incorporate jumps and allow only for discrete trading?
- What is the impact of different fee structures on the return distribution?

Direction

We simulate the return distribution of different strategies in a displaced double-exponential jump diffusion model parametrized to resemble the daily log returns of the MSCI World index for the last thirty years [3]

Jump risk



Outline

Outline

Model equation equity

The model equation:

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_{t-}} &= \mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t + \mathrm{d}\left(\sum_{j=1}^{N_t}(V_j - 1)\right) \\ S_T &= S_t \exp\left[\left(\mu - \frac{\sigma^2}{2} - \delta\right)\tau + \sigma W_{T-t}\right]\prod_{j=1}^{N_{T-t}} V_j \end{aligned}$$

Model equation equity

The model equation:

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}\left(\sum_{j=1}^{N_t}(V_j - 1)\right)$$

$$S_T = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2} - \delta\right)\tau + \sigma W_{T-t}\right]\prod_{j=1}^{N_{T-t}} V_j$$

- (W_t) standard Brownian motion,
- (N_t) Poisson process with intensity $\lambda > 0$
- V_j i.i.d. $V_j \sim e^Y$: Y represents the relative jump size with a minimal jump of κ, therefore leading to jumps of Y in the range $(-\infty, -\kappa] \cup [\kappa, +\infty)$,
- δ drift adjustment $\Rightarrow S_t$ has the desired drift μ

Model equation equity

The model equation:

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}\left(\sum_{j=1}^{N_t}(V_j - 1)\right)$$

$$S_T = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2} - \delta\right)\tau + \sigma W_{T-t}\right]\prod_{j=1}^{N_{T-t}} V_j$$

- (W_t) standard Brownian motion,
- (N_t) Poisson process with intensity $\lambda > 0$
- V_j i.i.d. $V_j \sim e^Y$: Y represents the relative jump size with a minimal jump of κ, therefore leading to jumps of Y in the range $(-\infty, -\kappa] \cup [\kappa, +\infty)$,
- δ drift adjustment $\Rightarrow S_t$ has the desired drift μ

The processes (W_t), (N_t), and the random variables V_j are all independent.

Jumpsize

We chose the jumps Y to be exponentially distributed outside $(-\kappa, +\kappa)$.

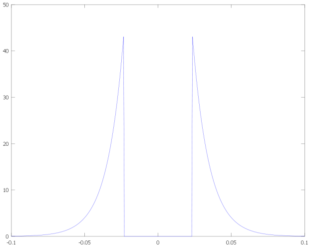


Figure: Displaced Double-Exponential density of Y with parameters $\kappa = 2.31\%$, $\eta_1 = \eta_2 = \eta = 1/1.121\%$, $p = 0.5$

Drift adjustment

Drift adjustment as in Kou [4]:

$$\begin{aligned}\delta &= \boldsymbol{E}[e^{Y} - 1] \\ &= \lambda\left(p\eta_1 \frac{e^{+\kappa}}{\eta_1 - 1} + (1-p)\eta_2 \frac{e^{-\kappa}}{\eta_2 + 1} - 1\right).\end{aligned}$$

We use $\eta = \eta_1 = \eta_2$ and $p = 0.5$.

Moments

First moment:

$$
\begin{aligned}
& \boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right] \\
= \quad & \sum_{n=0}^{\infty} \boldsymbol{E}\left[\sum_{k=1}^{n} U_k(\kappa + H_k)\right] \cdot \boldsymbol{P}[N_t = n] \\
= \quad & \sum_{n=0}^{\infty} n \cdot 0 \cdot \boldsymbol{P}[N_t = n] \\
= \quad & 0
\end{aligned}
$$

Moments

First moment:

$$\begin{aligned} & \boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right] \\ = & \sum_{n=0}^{\infty} \boldsymbol{E}\left[\sum_{k=1}^{n} U_k(\kappa + H_k)\right] \cdot \boldsymbol{P}[N_t = n] \\ = & \sum_{n=0}^{\infty} n \cdot 0 \cdot \boldsymbol{P}[N_t = n] \\ = & \ 0 \end{aligned}$$

H_k: independent exponentially distributed with expectation $h = \frac{1}{\eta}$

Moments

First moment:

$$
\begin{aligned}
& \boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right] \\
= & \sum_{n=0}^{\infty} \boldsymbol{E}\left[\sum_{k=1}^{n} U_k(\kappa + H_k)\right] \cdot \boldsymbol{P}[N_t = n] \\
= & \sum_{n=0}^{\infty} n \cdot 0 \cdot \boldsymbol{P}[N_t = n] \\
= & \; 0
\end{aligned}
$$

H_k: independent exponentially distributed with expectation $h = \frac{1}{\eta}$
U_k: $+1$ and -1 with probability $\frac{1}{2}$

Second moment:

$$
\begin{aligned}
& \boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right]^2 \\
= & \sum_{n=0}^{\infty} \boldsymbol{E}\left[\sum_{k=1}^{n} U_k(\kappa + H_k)\right]^2 \cdot \boldsymbol{P}[N_t = n] \\
= & \lambda t((\kappa + h)^2 + h^2)
\end{aligned}
$$

Variance

Variance:

$$
\begin{aligned}
& \mathbf{var}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right] \\
= \quad & \boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right]^2 - \left(\boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right]\right)^2 \\
= \quad & \lambda t((\kappa + h)^2 + h^2)
\end{aligned}
$$

Variance

Variance:

$$
\begin{aligned}
& \mathbf{var}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right] \\
= \quad & \boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right]^2 - \left(\boldsymbol{E}\left[\sum_{k=1}^{N_t} U_k(\kappa + H_k)\right]\right)^2 \\
= \quad & \lambda t((\kappa + h)^2 + h^2)
\end{aligned}
$$

Volatility of the DDE-process:

$$
\sqrt{\frac{1}{t}\mathbf{var}\left[\ln \frac{S_t}{S_0}\right]} \quad = \quad \sqrt{\sigma^2 + \lambda((\kappa + h)^2 + h^2)}
$$

Parameter estimation

- data: **MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD** January 1 1980 to October 2 2009.

Parameter estimation

- data: **MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD** January 1 1980 to October 2 2009.
- prices: $x_0, x_1, \ldots, x_N$

Parameter estimation

- data: **MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD** January 1 1980 to October 2 2009.
- prices: $x_0, x_1, \ldots, x_N$
- log-returns:

$$r_i \triangleq \ln \frac{x_i}{x_{i-1}}, \quad i = 0, 1, \ldots, N \tag{1}$$

Parameter estimation

- data: **MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD** January 1 1980 to October 2 2009.
- prices: $x_0, x_1, \ldots, x_N$
- log-returns:

$$r_i \stackrel{\Delta}{=} \ln \frac{x_i}{x_{i-1}}, \quad i = 0, 1, \ldots, N \tag{1}$$

- estimate for the daily log return:

$$\overline{r} = \frac{1}{N} \sum_{i=1}^{N} r_i \tag{2}$$

Parameter estimation

- data: **MSCI Daily TR (Total Return) Gross (gross dividends reinvested) in USD** January 1 1980 to October 2 2009.
- prices: $x_0, x_1, \ldots, x_N$
- log-returns:

$$r_i \stackrel{\Delta}{=} \ln \frac{x_i}{x_{i-1}}, \quad i = 0, 1, \ldots, N \tag{1}$$

- estimate for the daily log return:

$$\overline{r} = \frac{1}{N} \sum_{i=1}^{N} r_i \tag{2}$$

- estimate for the total volatility $\widehat{\sigma}_{tot}$:

$$\widehat{\sigma}_{tot}^2 = \frac{\#\text{Prices per year}}{N-1} \left(\sum_{i=1}^{N} r_i^2 - N\overline{r}^2 \right) \tag{3}$$

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0, 1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0, 1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.
- The level u should be chosen such that the resulting returns are intuitively considered as jumps.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0,1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.
- The level u should be chosen such that the resulting returns are intuitively considered as jumps.
- If u is chosen too high, even small log-returns are considered jumps.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0, 1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.
- The level u should be chosen such that the resulting returns are intuitively considered as jumps.
- If u is chosen too high, even small log-returns are considered jumps.
- If u is too low, almost no jumps occur.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0, 1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.
- The level u should be chosen such that the resulting returns are intuitively considered as jumps.
- If u is chosen too high, even small log-returns are considered jumps.
- If u is too low, almost no jumps occur.
- Of course, this level is subjective.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0,1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.
- The level u should be chosen such that the resulting returns are intuitively considered as jumps.
- If u is chosen too high, even small log-returns are considered jumps.
- If u is too low, almost no jumps occur.
- Of course, this level is subjective.
- We have chosen $u = 1\%$ because in this case only daily changes of more than 2% are considered to be a jump.

Parameter estimation

- To determine the parameters for the jump process we have to define a level κ such that r_i with $\|r_i\| \geq \kappa$ is considered a jump.
- To determine this κ we define for a given level $u \in [0, 1]$ the $u\%$ smallest and $u\%$ biggest daily log returns.
- The level u should be chosen such that the resulting returns are intuitively considered as jumps.
- If u is chosen too high, even small log-returns are considered jumps.
- If u is too low, almost no jumps occur.
- Of course, this level is subjective.
- We have chosen $u = 1\%$ because in this case only daily changes of more than 2% are considered to be a jump.
- Changes of less than 2% can be explained with the diffusion part with sufficiently high probability.

Parameter estimation

- For the analyzed MSCI World Index, the smallest up-jump and the smallest down-jump are almost equal.

Parameter estimation

- For the analyzed MSCI World Index, the smallest up-jump and the smallest down-jump are almost equal.
- The absolute values of the relative jump size is on average 2.31%. We use the 2% quantile of the absolute relative jump size to estimate κ.

Parameter estimation

- For the analyzed MSCI World Index, the smallest up-jump and the smallest down-jump are almost equal.
- The absolute values of the relative jump size is on average 2.31%. We use the 2% quantile of the absolute relative jump size to estimate κ.
- The value η of the single parameter exponential distribution is chosen such that η fits the mean of the observed jumps.

Parameter estimation

- For the analyzed MSCI World Index, the smallest up-jump and the smallest down-jump are almost equal.
- The absolute values of the relative jump size is on average 2.31%. We use the 2% quantile of the absolute relative jump size to estimate κ.
- The value η of the single parameter exponential distribution is chosen such that η fits the mean of the observed jumps.
- From the financial data and the already fixed parameters we obtain $h = 1/\eta$ with $\eta = 1.21\%$.

Parameter estimation

- For the analyzed MSCI World Index, the smallest up-jump and the smallest down-jump are almost equal.
- The absolute values of the relative jump size is on average 2.31%. We use the 2% quantile of the absolute relative jump size to estimate κ.
- The value η of the single parameter exponential distribution is chosen such that η fits the mean of the observed jumps.
- From the financial data and the already fixed parameters we obtain $h = 1/\eta$ with $\eta = 1.21\%$.
- The number of jumps divided by the total number of observations yields an estimate for the jump frequency. Annualizing this frequency we can estimate λ to be 5.21.

Parameter estimation

- For the analyzed MSCI World Index, the smallest up-jump and the smallest down-jump are almost equal.
- The absolute values of the relative jump size is on average 2.31%. We use the 2% quantile of the absolute relative jump size to estimate κ.
- The value η of the single parameter exponential distribution is chosen such that η fits the mean of the observed jumps.
- From the financial data and the already fixed parameters we obtain $h = 1/\eta$ with $\eta = 1.21\%$.
- The number of jumps divided by the total number of observations yields an estimate for the jump frequency. Annualizing this frequency we can estimate λ to be 5.21.
- Finally we have to correct the estimator for the volatility according to Equation (1) since the volatility consists of the jump part and the diffusion part.

Parameter estimation

Parameter	Value
Total volatility $\widehat{\sigma}_{tot}$	14.3 %
Volatility of the diffusion part $\widehat{\sigma}$	11.69%
Jump intensity λ	5.209
Minimum jump size κ	2.31%
Expected jump size h above minimum jump size	1.21%
Drift adjustment δ	0.339%

Table: Estimated parameters for the DDE-process.

Interest rate model

Zero bond curve

To calculate the current value of the future liability (floor) and the performance of the riskless investments, we model the short rate by a Hull-White Extended Vasicek Model. The model is calibrated to the zero bond curve as of October 1 2009. The curve is extracted from the money market and swap rate quotes on Reuters.

The model equation:

$$\mathrm{d}r_t = [\theta_t - ar_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t$$

with constants a and σ and time dependent ϑ_t chosen to exactly fit the term structure of interest rates.

Outline

Classical insurance strategy

- In this strategy a large proportion of the invested capital is held in the actuarial reserve fund to fully generate the guarantee.
- Only the remaining capital is invested in products with a higher equity proportion.
- The actuarial reserve fund is assumed to be riskless and accrues the interest implied by the current zero bond curve.
- It guarantees a minimum yearly interest rate of 2.25%.
- We assume that the calculation of the amount needed to meet the future liability is based on the guaranteed interest rate of 2.25%.

Classical insurance strategy

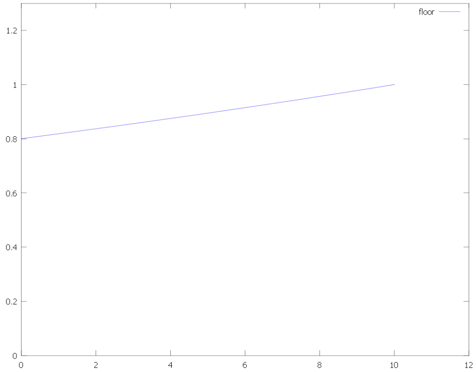


Figure: Simulated path for a classical insurance strategy and 10 year investment horizon

Classical insurance strategy

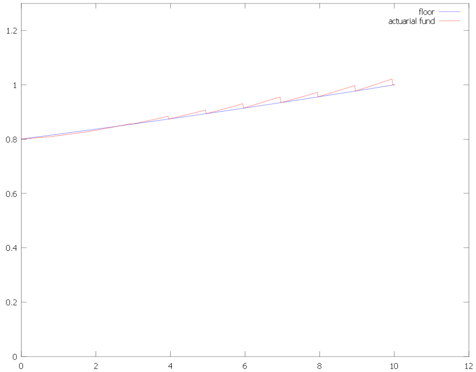


Figure: Simulated path for a classical insurance strategy and 10 year investment horizon

Classical insurance strategy

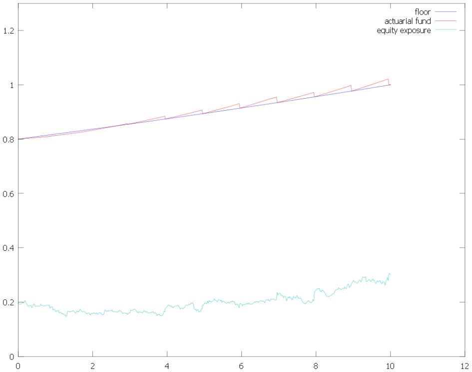


Figure: Simulated path for a classical insurance strategy and 10 year investment horizon

Classical insurance strategy

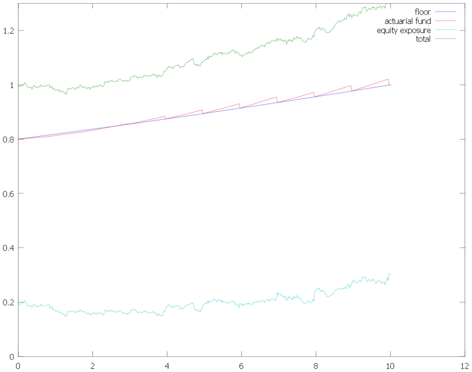


Figure: Simulated path for a classical insurance strategy and 10 year investment horizon

Classical insurance strategy

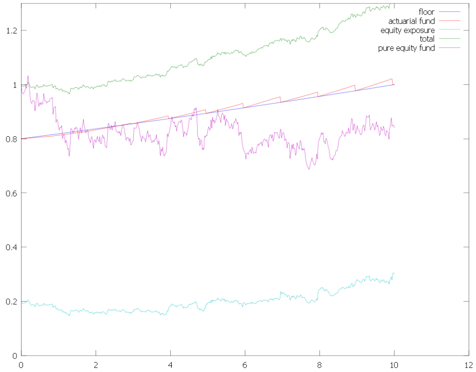


Figure: Simulated path for a classical insurance strategy and 10 year investment horizon

Constant proportion portfolio insurance

- In contrast to the traditional strategy the amount necessary to generate the guarantee is not fully invested in the riskless products.
- The amount invested in the more risky equity products is leveraged for a higher equity exposure.
- Continuous monitoring ensures that the guarantee is not at risk, since the equity proportion is reduced with the portfolio value becoming closer to the floor.
- If the process allows for jumps or if trading is done only at discrete time points we are again imposed to gap risk.

With F being the floor of the future obligations, NAV the net asset value of the fund and a the leverage factor, the rebalancing equation for the risky asset R is

$$R = \max\left(a\left(NAV - F\right), NAV\right). \tag{4}$$

Constant proportion portfolio insurance

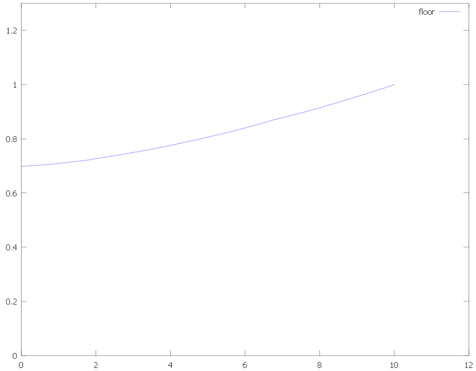


Figure: Simulated CPPI path with leverage factor 3 and 10 years investment horizon

Constant proportion portfolio insurance

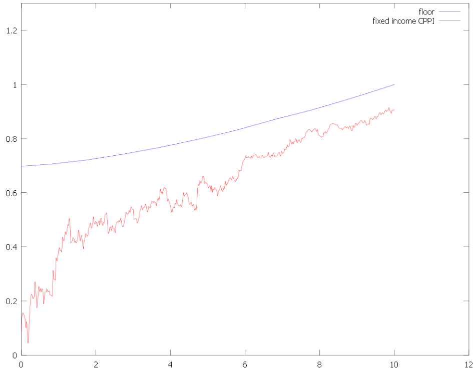


Figure: Simulated CPPI path with leverage factor 3 and 10 years investment horizon

Constant proportion portfolio insurance

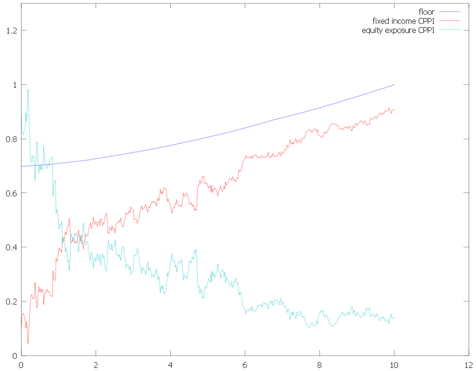


Figure: Simulated CPPI path with leverage factor 3 and 10 years investment horizon

Constant proportion portfolio insurance

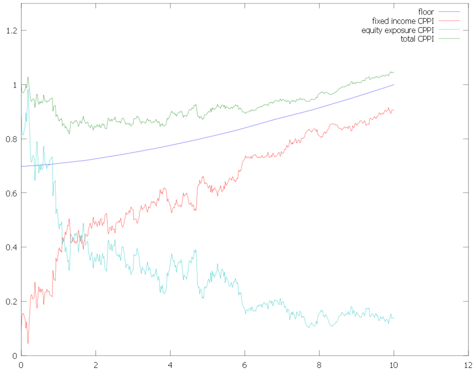


Figure: Simulated CPPI path with leverage factor 3 and 10 years investment horizon

Constant proportion portfolio insurance

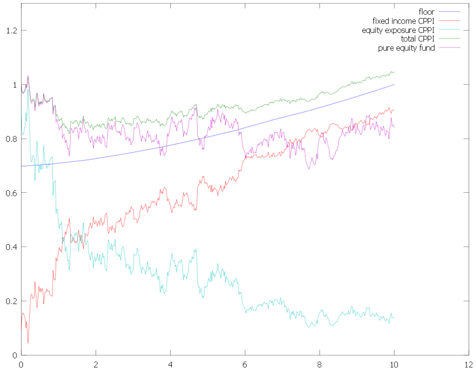


Figure: Simulated CPPI path with leverage factor 3 and 10 years investment horizon

Stop loss strategy

- 100% of the equity amount is held in the risky fund until the floor is reached. In this case all the investment is moved to the fixed income fund to generate the guarantee at maturity.
- The Strategy is riskless as the CPPI strategy in a continuous model. In a model with jumps or if trading is only at discrete time points again we are imposed to gap risk.
- We neglect liquidity issues here which actually forces the insurer to liquidate the risky asset before it reaches the floor level.

Stop loss strategy

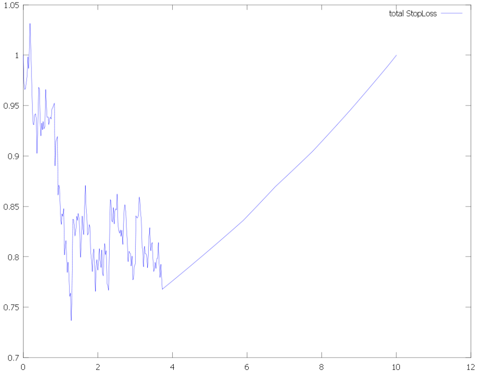


Figure: Simulated stop-loss path with 10 years investment horizon

Outline

Payments to the contract

- We consider a typical payment plan for a Riester-Rente with an horizon of 20 years and a monthly payment of 100 Euro (sum of own payments and state payments).

Payments to the contract

- We consider a typical payment plan for a Riester-Rente with an horizon of 20 years and a monthly payment of 100 Euro (sum of own payments and state payments).
- We further assume that the investor has one child born after January 1 2008 and earns 30,000 Euro per year. In this case the insurant receives 454 Euro from the state per year, so he actually only has to pay 746 Euro per year to reach 1,200 Euro per year. This is a very high support rate of 37%.

Payments to the contract

- We consider a typical payment plan for a Riester-Rente with an horizon of 20 years and a monthly payment of 100 Euro (sum of own payments and state payments).
- We further assume that the investor has one child born after January 1 2008 and earns 30,000 Euro per year. In this case the insurant receives 454 Euro from the state per year, so he actually only has to pay 746 Euro per year to reach 1,200 Euro per year. This is a very high support rate of 37%.
- For comparison, if we take an investor without children, earning 52,500 Euro per year, the support rate would only be maximal 7.3%.

Payments to the contract

- We consider a typical payment plan for a Riester-Rente with an horizon of 20 years and a monthly payment of 100 Euro (sum of own payments and state payments).
- We further assume that the investor has one child born after January 1 2008 and earns 30,000 Euro per year. In this case the insurant receives 454 Euro from the state per year, so he actually only has to pay 746 Euro per year to reach 1,200 Euro per year. This is a very high support rate of 37%.
- For comparison, if we take an investor without children, earning 52,500 Euro per year, the support rate would only be maximal 7.3%.
- We consider different cost structures (α-cost,β-cost and capital management cost), all of them having the same current value but beeing differently distributed over time.

Outline

Return distribution standard scenario (6% drift)

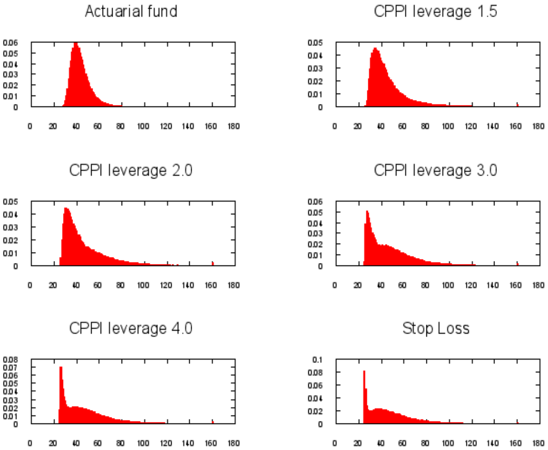


Figure: Return distribution of the different strategies. We list the capital available at retirement (in units of 1,000 EUR) on the x-axes.

Results

Product	expected capital (constant rates)	exposure (constant rates)	expected capital (stochastic rates)	exposure (stochastic rates)
CPPI leverage factor 1,5	46,187	74.86%	45,731	74.17%
CPPI leverage factor 2	47,368	88.49%	46,927	87.41%
CPPI leverage factor 3	47,881	94.22%	47,618	93.51%
CPPI leverage factor 4	48,007	95.66%	47,840	95.13%
Stop Loss	48,190	97.12%	48,159	96.57%
Actuarial reserve fund	42,602	31.59%	44,294	31.50%

Table: Expected capital at retirement.

Impact of jumps

Product	number of paths with gap (IR constant)	average realized gap (IR constant)	number of paths with gap (IR stochastic)	average realized gap (IR stochastic)
Actuarial reserve	0	0	0	0
CPPI, factor 1,5	0	0	8	346
CPPI, factor 2	0	0	52	371
CPPI, factor 3	0	0	489	321
CPPI, factor 4	0	0	1,664	306
Stop Loss	18,642	223	18,902	441

Table: Shortfalls for 100,000 simulations.

Bibliography

Black, F. and Scholes, M. (1973).
The pricing of Options and Corporate Liabilities, *Journal of Political Economy*, (81)

Detering, N., Weber, A. and Wystup, U. (2009).
Riesterrente im Vergleich - Eine Simulationsstudie zur Verteilung der Rendite im Auftrag von Euro-Magazin, *MathFinance Research Paper*

Detering, N., Weber, A. and Wystup, U. (2013).
Return distributions of equity-linked retirement plans under jump and interest rate risk, European Actuarial Journal, Volume 3, Issue 1 (2013), pp. 203-228, June 2013

Kou, S.G. (2002).
A Jump-Diffusion-Model for option pricing. *Management Science*, **48**, (8)

More information about MathFinance can be found on
https://www.mathfinance.com/