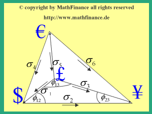

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FX exotics and the relevance of computational methods in their pricing and risk management

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Abstract

Starting with an overview of the current FX derivatives industry we take a look at a few examples where computational methods are crucial to run the daily business. The examples will include instalment contracts, accumulative forward contracts and the efficient computation of option price sensitivities

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1. Overview

EUR/USD is one of the most liquid underlying markets
Trading activities in FX are

1. Spot/Forward (90%) - extremely small margins
2. Vanilla Options (9%) - small margins
3. Exotic Options (1%) - potentially higher margins

1.1. FX Exotics

1. barrier and touch options
2. compound and instalment
3. average rate options
4. forward start and cliquets
5. corridors/fader/accumulative options
6. quanto options
7. multi-currency options: baskets, bestof, outside barriers
8. vol- and variance swaps
9. structured products

2. Accumulative Forward

Market of Jan 7 2003, EUR/USD Spot at $S_0 = 1.0200$. Zero cost contract for $T = 1$ year.

Client sells 200k USD at $K = 0.9700$ every day the EUR/USD fixing F_{t_i} is between $K = 0.9700$ and $B = 1.0700$.

Client sells 400k USD at $K = 0.9700$ every day the EUR/USD fixing F_{t_i} is below $K = 0.9700$.

If $B = 1.0700$ ever trades, then the client stops accumulating but keeps 50% of the accumulated amount.

Total of 255 Fixings.

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Payoff per 200k USD is

$$
\begin{aligned}
& (S_T - K) \sum I\!I_{\{F_{t_i} < B\}} \left[50\% I\!I_{\{S_t < B \forall t\}} + 50\% I\!I_{\{t_i < \tau\}} \right] \\
+ \; & (S_T - K) \sum I\!I_{\{F_{t_i} < K\}} \left[50\% I\!I_{\{S_t < B \forall t\}} + 50\% I\!I_{\{t_i < \tau\}} \right], \\
& \tau \stackrel{\Delta}{=} \inf\{t : S_t \geq B\}. \qquad (1)
\end{aligned}
$$

TV can be computed in closed form (see [7]).

What is the market price?

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2.1. Pricing and Hedging: Method 1

replicate the structure using options we can price over TV

A Client buys strip of 0.9700 eur call , RKO 1.07. We price the 3,6,9,12 month

month	*bp*
3	+50
6	+35
9	+30
12	+23

Average of 34 bp over for nominal amount of 255 * 200,000 / 0.9700 = 52.58 MIO EUR

Overhedge A = 179 K

B Client sells strip of 0.9700 eur put , KO 1.07. We price the 3,6,9,12 month

month	*bp*
3	-5
6	-15
9	-20
12	-20

Average of 15 bp under for nominal amount of 255 * 400,000 / 0.9700 = 105.15 MIO EUR

Overhedge B = 158 K

C Client sells a one-touch 1.0700 (to account for the 50% reduction of his payout if we touch 1.0700) maturity 1 year.
Price is 4% under TV.
Payoff of the one-touch = 50% * 50 mio * (1.07-0.97) = 2.5 MIO
Overhedge C = 2.5 mio * 4% = 100 K

Total Overhedge = A + B + C = 437 K

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2.2. Pricing and Hedging: Method 2

Looking at the cost of vega management

A structure has 25K negative *volga* ... cost 115K (using a butterfly)

B structure has 325K negative *vanna* between 0.99 and 1.09 ... cost 285K using the price of a 1 year Risk Reversal

C structure has 200K of vega ... cost 20 K of spread (0.1 vol versus mid- market)

Total Overhedge = A + B + C = 420 K EUR

2.3. Financial Engineering Issues

1. Need fast calculators for TVs, ideally closed-form solutions
2. Automate computation of the hedge and its cost
3. Live market data feed: Spot, Termstructure of Interest Rates, Vol-surface
4. For Method 1: shift exotic risk to liquid risk, i.e. using first generation exotics to price 2nd generation exotics
5. For Method 2: shift exotic risk to liquid risk, i.e. using vanillas to price first generation exotics.

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2.4. Pricing a one-touch

- pays a fixed amount of a pre-specified currency, if the underlying ever touches a barrier
- costs between 0% and 100%
- the closer the spot at the barrier, the more expensive the one-touch
- market price often far away from TV, due to cost of risk management

All details in [17].

Example for market: EUR/USD 17 July 2002 1.0045 EUR 3.33% USD 1.76%, 3 M ATM vol 11.85%, RR 1.25%, BF 0.25%

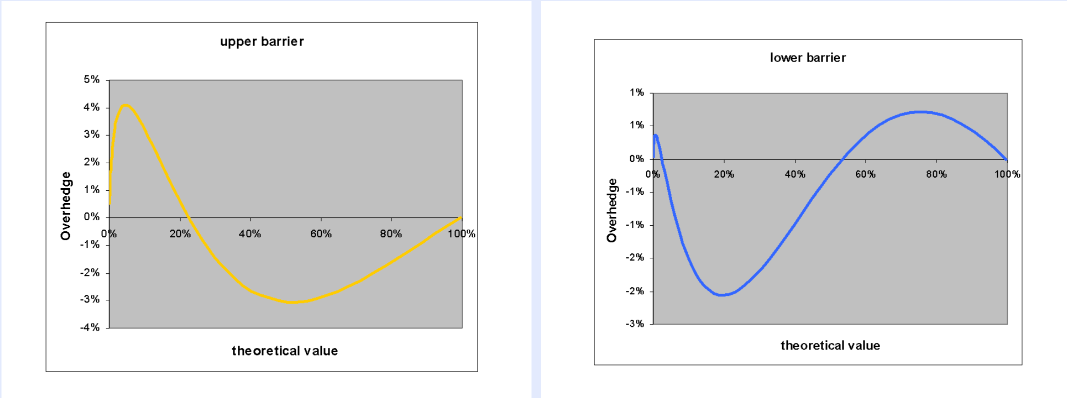

Figure 1: Overhedge for one-touch options

The Overhedge calculation

- Market price of the option
- $=$ TV (theoretical value)
- $+p\cdot$ vanna of the option $\cdot$ value RR / vanna RR
- $+p\cdot$ volga of the option $\cdot$ value BF / volga BF

where

- RR: Risk Reversal
- BF: Butterfly
- p: probability that the hedge is needed

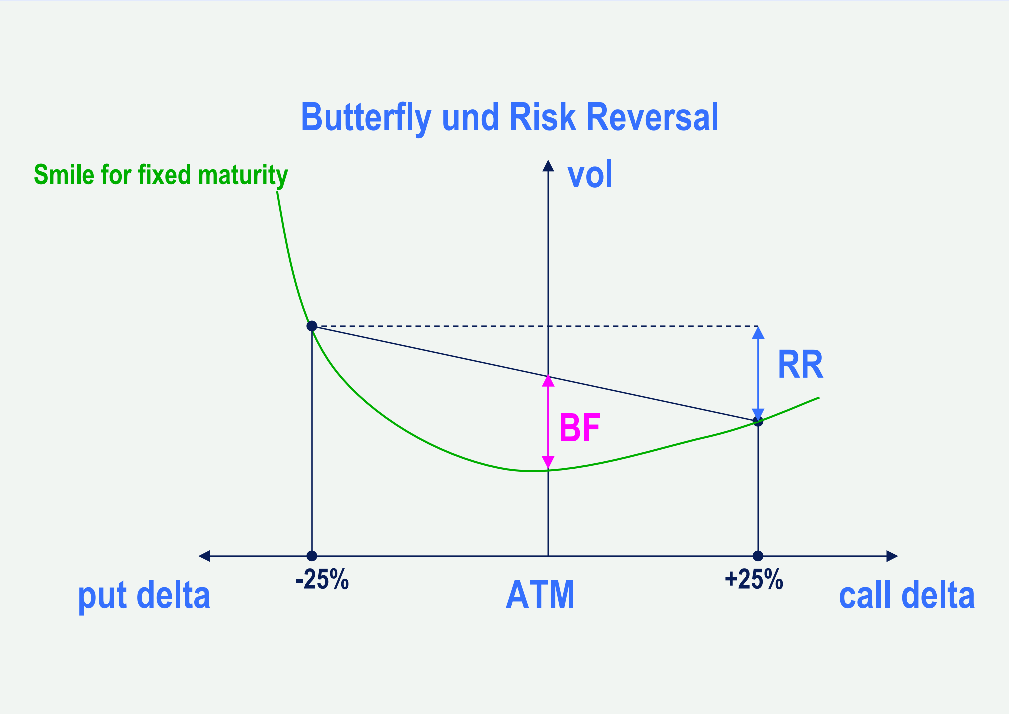

Figure 2: Butterfly and Risk Reversal

Example

- 1Y USD/JPY one-touch at 127.00, notional in USD
- Market data: 117.00 spot, 8.80% vol, 2.10% USD interest rate, 0.10% JPY interest rate, 25delta RR -0.45%, 25delta BF 0.37%
- TV: 38.2%, Vanna -9.0, Volga -1.0

Market price is computed as TV = 38.2%

- $+p\cdot$ -9.0 $\cdot$ -0.15% / 4.5
- $+p\cdot$ -1.0 $\cdot$ 0.27% /0.035
- $= 38.2\% + p \cdot [0.3\% - 7.7\%] = 38.2\% - p \cdot 7.4\%$

where

- $p = 100\% - 38\%$
- so, overhedge is $62\% \cdot -7.4\% = -4.7\%$
- so, market mid price is $38.2\% - 4.7\% = 33.5\%$
- so, bid - ask could be 32%/35%
- and the hedge: sell 2 RR and 28 BF

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3. Instalment Options

Joint work with Susanne Griebsch, Goethe University.

3.1. What is an Instalment Option?

- Like Vanilla Option, but

  (1) Premium is divided into several payments and is paid periodically on so-called "instalment dates"

  (2) Holder has the right to cancel option through the termination of instalment payments

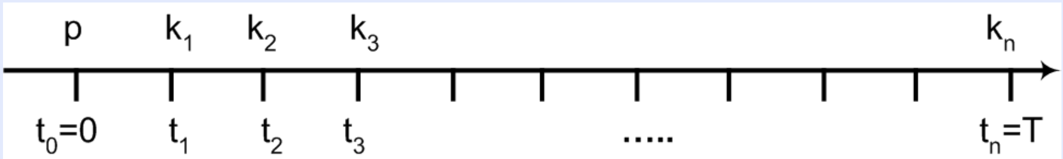

Figure 3: Dates for Instalment Payments

- Other names: continuation option, pay-as-you-go option, a generalization of compound option

- n-Instalment Option can be understood as a series of n options depending on each other

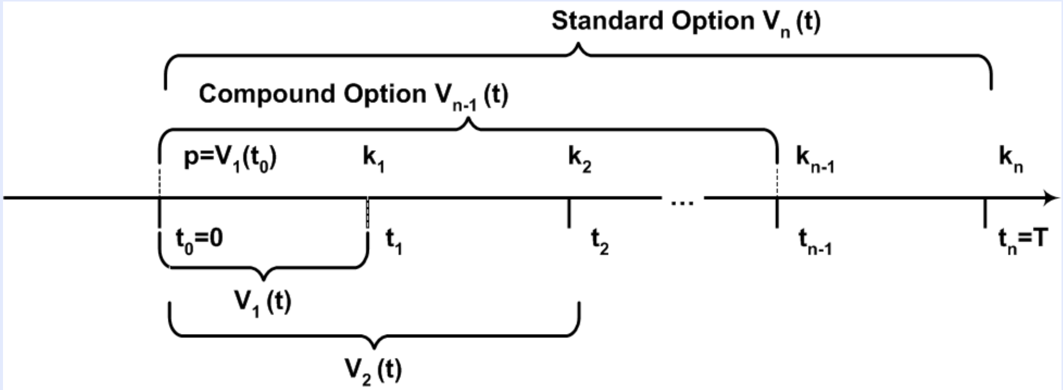

Figure 4: Lifetimes of the options V_i

- Characterized by
  - n exercise times $t_1, ..., t_n = T$ (often $t_i = iT/n$ for all i),
  - n strike prices $k_1, ..., k_n$,
  - n put/ call indicators $\phi_1, ..., \phi_n$ where $\phi_i := \begin{cases} +1 & \text{if option } i \text{ is a call} \\ -1 & \text{if option } i \text{ is a put} \end{cases}$

Market data

- S_0: spot
- r_d: domestic interest rate
- r_f: foreign interest rate
- σ: volatility

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3.2. Advantages of Instalment Options

- Traded over-the-counter tailor-made to client needs
- Prevention of losses through possibility of termination
- Helpful in situations where necessity of hedge is uncertain
- Low initial premium is easy to schedule in the firm's budget

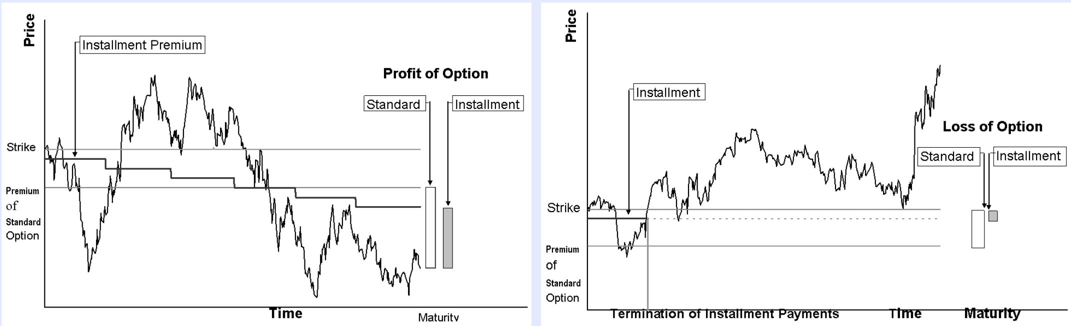

Figure 5: Comparison of Instalment Option with Vanilla Put: Continuation of instalment payments until expiration vs. Continuation of instalment payments until expiration

3.3. Example of a Traded Instalment Option

- Application area: International Treasury Management
- Corporate buys EUR Call/ USD Put 25 Mio EUR notional
- Strike price: 1.0500 EUR/USD
- Exercise type: European
- Maturity date: 17 Dec 2003, Delivery settlement on 19 Dec 2003
- Transaction date: 19 Dec 2002
- EUR USD spot ref: 1.0259
- Premium and strike prices: 285,500.00 USD
- Decision and Value dates: 31/03/03, 02/04/03, 30/06/03, 02/07/03, 30/09/03, 02/10/03
- The corporate has extended the instalment at all dates and finally sold the EUR call on Nov 19 2003 for a profit of 2.77 MIO EUR (spot was at 1.1900).

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3.4. Pricing of Instalment Options in the Black-Scholes Model

- Like Vanilla Options or Compound Options, i.e. discounted expectation of payoff function

- $dS_t = S_t[(r_d - r_f)dt + \sigma dW_t]$ for $0 \leq t \leq T$
  $S_{t_2} = S_{t_1} \exp((r_d - r_f - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}Z)$, for $0 \leq t_1 \leq t_2 \leq T$),
  $\Delta t = t_2 - t_1$

- Payoff at maturity is $\max(\phi_n(S_T - k_n), 0) \stackrel{def}{=} (\phi_n(S_T - k_n))^+$

- Date before last instalment date t_{n-1} buyer pays k_{n-1} to receive classical european option, in which the price at t_{n-1} is described by

$$V_n(s) \stackrel{def}{=} V_{Std}(s) = e^{-r_d(t_n - t_{n-1})}\mathbb{E}[\phi_n[S_T - k_n]^+ \mid S_{t_{n-1}} = s]$$

- Rational buyer only pays instalment rate if $V_{Std} \geq k_{n-1}$
  shortly before instalment date option is worth $\max(V_{Std} - k_{n-1}, 0)$

- Compound option price at time t_{n-2} is

$$V_{n-1}(s) \stackrel{def}{=} V_{Cp}(s) = e^{-r_d(t_{n-1} - t_{n-2})}\mathbb{E}[\phi_{n-1}[V_n - k_{n-1}]^+ \mid S_{t_{n-2}} = s]$$

- Next steps are analogous, compound option V_i with option V_{i+1} so that V_i is an option on V_{i+1} with strike k_i and decision date t_i

- Exact expression for value function of Instalment Option

  $$V_i(s) \stackrel{def}{=} e^{-r_d(t_i - t_{i-1})}\mathbb{E}[(\phi_i(V_{i+1}(t_{i+1}) - k_i))^+ \mid S_{i-1} = s], \text{ for } i = 1, ..., n-1.$$

- When carried out for all $i \leq n-1$, result is first instalment which is paid to open the deal at $t_0 = 0$

  $$p \stackrel{def}{=} V_1(s) = e^{-r_d(t_1 - t_0)}\mathbb{E}[\phi_1[V_2 - k_1]^+ \mid S_{t_0} = s]$$

- Nested expectations require analysis of multiple integrals

- Numerical computation of multiple integrals is time consuming and possibly imprecise

3.5. n-variate Cumulative Normal Formula

- n-variate cumulative normal function

$$N_n(h_i;\{\rho_{ij}\}_{1\le j\le n, i<j}) = \text{Prob}\{Z_i < h_i; i = 1, ..., n\} = \int_{-\infty}^{h_1} ... \int_{-\infty}^{h_n} n(x_1, ..., x_n) dx_n ... dx_1$$

- Curnow and Dunett (1962), see [5], show

$$N_n(h_i;\{\rho_{ij}\}) = \int_{-\infty}^{h_1} N_{n-1}\left(\frac{h_i - \rho_{i1}y}{(1-\delta_{i1}^2)^{\frac{1}{2}}};\{\rho_{ij*1}\}\right) n(y)dy \qquad i = 2, ..., n$$

$$\rho_{ij*1} = \frac{\rho_{ij} - \rho_{i1}\rho_{j1}}{(1-\delta_{i1}^2)^{\frac{1}{2}}(1-\delta_{j1}^2)^{\frac{1}{2}}} \quad (i, j \ne 1 \text{ and } j \ne i)$$

- Special case $n = 2$ was used for compound option formula

$$N_2(h_1, h_2; \rho) = \int_{-\infty}^{h_1} N\left(\frac{h_2 - \rho y}{(1-\rho^2)^{\frac{1}{2}}}\right) n(y)dy$$

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$$
\begin{aligned}
V_{Cp} &= e^{r_f t_2} S_0 N_2 \left[\frac{\ln \frac{S_0}{S_1} + \mu^{(+)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2} + \mu^{(+)} t_2}{\sigma \sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}} \right] \\
&- e^{-r_d t_2} k_1 N_2 \left[\frac{\ln \frac{S_0}{S_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}} \right] \\
&- e^{-r_d t_1} k_2 N \left[\frac{\ln \frac{S_0}{S_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}} \right]
\end{aligned}
$$

n-variate case

- $\vec{k} = (k_1, ..., k_n)$ strike prices
- $\vec{t} = (t_1, ..., t_n)$ instalment dates
- $\vec{\phi} = (\phi_1, ..., \phi_n)$ put/call indicators
- correlation coefficients of n-variate cumulative normal functions

$$\rho_{ij} = \sqrt{t_i / t_j} \text{ for } i, j = 1, ..., n \text{ and } i < j$$

$$
\begin{aligned}
& V_n(S_0, \vec{k}, \vec{t}, \sigma, r_d, r_f, \vec{\phi}) \\
= \quad & e^{-r_f t_n} S_0 \phi_1 \cdot \ldots \cdot \phi_n \\
& N_n \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(+)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(+)} t_2}{\sigma \sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{\bar{S}_n} + \mu^{(+)} t_n}{\sigma \sqrt{t_n}}; \{\rho_{ij}\} \right] \\
- \quad & e^{-r_d t_n} k_n \phi_1 \cdot \ldots \cdot \phi_n \\
& N_n \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{\bar{S}_n} + \mu^{(-)} t_n}{\sigma \sqrt{t_n}}; \{\rho_{ij}\} \right] \\
- \quad & e^{-r_d t_{n-1}} k_{n-1} \phi_2 \cdot \ldots \cdot \phi_n \\
& N_{n-1} \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{\bar{S}_{n-1}} + \mu^{(-)} t_{n-1}}{\sigma \sqrt{t_{n-1}}}; \{\rho_{ij}\} \right] \\
& \vdots \\
- \quad & e^{-r_d t_2} k_2 \phi_{n-1} \phi_n N_2 \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}; \rho_{12} \right] \\
- \quad & e^{-r_d t_1} k_1 \phi_n N \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}} \right]
\end{aligned}
$$

3.6. Binomial Tree Option Pricing Technique

- Binomial model was developed by Cox, Ross and Rubinstein
- Price movements of log-returns of underlying are modeled as constant up and down movements $(u = \exp(\sigma\sqrt{T/m}, d = \exp(-\sigma\sqrt{T/m})$ in the tree.

3.7. Algorithm for Pricing Instalment Options by H. Ben-Ameur, M. Breton and P. Fraincois [2]

- Approximation of value of Instalment Option at t_0 through piecewise linear interpolation, therefore solving dynamic programming equation which results in a closed form
- Exercise value is $V_n(s) = \max(0, \phi_n(S_T - k_n))$
- Holding value at t_i is $V_i^h(s) = \mathbb{E}[e^{-r_d \Delta t} V_{i+1}(S_{t_{i+1}}) \mid S_{t_i} = s]$ for $i = 0, ..., n-1$
  where

$$v_i(s) = \begin{cases} V_0^h(s) & \text{for } i = 0 \\ \max(0, V_i^h(s) - k_i) & \text{for } i = 1, ..., n-1 \\ V_n(s) & \text{for } i = n \end{cases}$$

- Net holding value $V_i^h(s) - k_i$

- $a_0 = 0 < a_1 < \ldots < a_p < a_{p+1} = +\infty$ set of points $R_0, \ldots, R_p$ partition of $\mathbb{R}^+$ in $(p+1)$ intervals $R_j = (a_j, a_{j+1}]$ for $j = 0, \ldots, p$
- Given approximations $\tilde{v}_i$ of option value v_i at a_j in step i piecewise linear interpolation of this function achieved through

$$\hat{v}_i(s) = \sum_{i=0}^{p} (\alpha_j^i + \beta_j^i s) I_{a_j < s \leq a_{j+1}}, \quad \tilde{v}_i(a_j) = \hat{v}_i(a_j), \quad \text{for } j = 0, \ldots, p-1,$$

$$\text{for } j = p \text{ choose } \alpha_p^i = \alpha_{p-1}^i \text{ and } \beta_p^i = \beta_{p-1}^i$$

- Assuming $\hat{v}_{i+1}$ is known, calculate expectation in step i

$$\begin{aligned} \tilde{v}_i^h(a_k) &= \mathbb{E}[e^{-r_d \Delta t} \hat{v}_{i+1}(S_{t_{i+1}}) | S_{t_i} = a_k] \\ &= e^{-r_d \Delta t} \sum_{j=0}^{p} \alpha_j^{i+1} \mathbb{E}[I_{\frac{a_j}{a_k} < e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} \leq \frac{a_{j+1}}{a_k}}] \\ &+ \beta_j^{i+1} a_k \mathbb{E}[e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} I_{\frac{a_j}{a_k} < e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} \leq \frac{a_{j+1}}{a_k}}], \end{aligned}$$

$\mu = r_d - r_f - \sigma^2/2$, $\tilde{v}_i$ approximated holding value of Instalment Option

- For $k = 1, ..., p$ and $j = 0, ..., p$ first integrals

$$A_{k,j} = \mathbb{E}[I_{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t}z} \leq \frac{a_{j+1}}{a_k}}] = \begin{cases} N(x_{k,1}) & \text{for } j = 0 \\ N(x_{k,j+1}) - N(x_{k,j}) & \text{for } 1 \leq j \leq p-1 \\ 1 - N(x_{k,p}) & \text{for } j = p \end{cases}$$

$$\begin{aligned} B_{k,j} &= \mathbb{E}[a_k e^{\mu\Delta t + \sigma\sqrt{\Delta t}z} I_{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t}z} \leq \frac{a_{j+1}}{a_k}}] \\ &= \begin{cases} a_k N(x_{k,1} - \sigma\sqrt{\Delta t}) e^{(r_d - r_f)\Delta t} & \text{for } \quad j = 0 \\ a_k [N(x_{k,j+1} - \sigma\sqrt{\Delta t}) - N(x_{k,j} - \sigma\sqrt{\Delta t})] e^{(r_d - r_f)\Delta t} & \text{for } \quad 1 \leq j \leq p-1 \\ a_k [1 - N(x_{k,p} - \sigma\sqrt{\Delta t})] e^{(r_d - r_f)\Delta t} & \text{for } \quad j = p \end{cases} \end{aligned}$$

with $x_{k,j} = [\ln(a_j/a_k) - \mu\Delta t]/(\sigma\sqrt{\Delta t})$.

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Procedure

0. Calculate a_i
1. Calculate $\hat{v}_n(s)$ for all s
2. Calculate $\tilde{v}^h_{n-1}(a_k)$ for all k in closed form
3. Calculate $\tilde{v}_{n-1}(a_k)$ for all k
4. Calculate $\hat{v}_{n-1}(s)$ for all $s > 0$
5. Iterate these steps until $\hat{v}_1(s_0)$=Price of Instalment Option at time 0 is calculated

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3.8. Comparison of Accuracy and Speed

- Results of binomial trees oscillate strongly

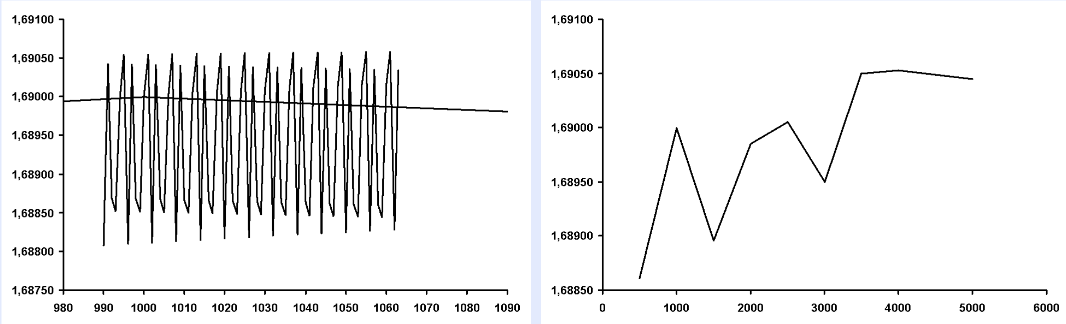

Figure 6: Convergence of the value function in the binomial trees implementation

- Trivariate formula is the fastest of all considered methods, even for higher numbers of instalments
- Accuracy of trivariate formula now only depends on accuracy of calculation of multivariate normal integrals and calculation of roots
- Algorithm of ABF works for equally distant instalment dates

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Performance

Numerical Method	**Value of V_{TV}**	**Time**
Binomial Trees $n = 4000$	$1,69053$	1109 sec
Trivariate Formula	$1,69092$	< 1 sec
Algorithm (Article of ABF) $p = 4000$	$1,69084$	168 sec
Numerical Int. (50000-point Gauss-Legendre)	$1,69087$	176 sec
Numer. Int. of Cp Formula (Mathematica)	$1,69091$	47 sec

Table 1: $S_0 = 100$, $k_1 = 100$, $k_{2,3} = 3$, $\sigma = 20\%$, $r_d = 10\%$, $r_f = 15\%$, $T = 1$, $\Delta t = 1/3$, $\phi_{1,2,3} = 1$

3.9. Convergence of Identical Premium

- Continuous Instalment Option is an american type option, where
  - Total sum of premiums paid at beginning
  - Difference repaid in case of an option termination
- Discounted sum of instalments

$$\underline{u}_n = f_n \sum_{i=0}^{n} e^{-r_d t_i} \qquad \text{where } t_i = (i-1)\Delta t \text{ and } n\Delta t = T$$

  $\underline{u}_n$ price of n-Instalment Option with instalment dates t_i and **identical premium** f_n paid at t_i, $0 \leq i \leq n-1$

- With increasing number of instalments n the total premium $\underline{u}_n$ increases (increasing optionality)
- With increasing n, instalment payments decrease
- $\underline{u}_n$ converges to an upper bound

$$U = g \int_0^T e^{-r_d s} ds$$

  $n \to \infty$ (and $\Delta t \to 0$)
  g is the uniform premium for continuous Instalment Option paid between gdt and $t + dt$ g corresponds with limit $\frac{f_n}{\Delta t} \to g$

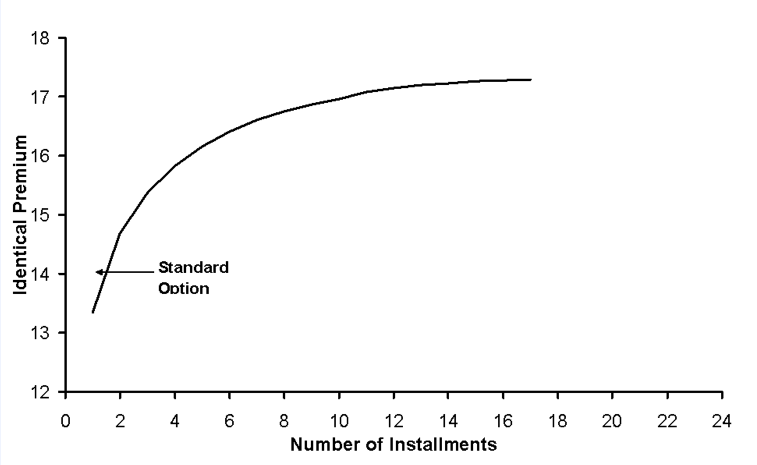

Figure 7: Convergence of uniform premium in discrete case to continuous premium

- How can we describe this upper bound?
- Possible approach:
  Continuous Instalment Option = Vanilla Call plus American Compound Put on this call with linearly decreasing strike (w.r.t. time)

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4. Greeks

Joint work with Oliver Reiss, Weierstrass Institute Berlin

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4.1. Notation

S	stock price or stock price process
B	cash bond, usually with risk free interest rate r
r	risk free interest rate
q	dividend yield (continuously paid)
σ	volatility of one stock, or volatility matrix of several stocks
ρ	correlation in the two-asset market model
t	date of evaluation ("today")
T	date of maturity
$\tau = T - t$	time to maturity of an option
x	stock price at time t
$f(\cdot)$	payoff function
$v(x, t, \ldots)$	value of an option
k	strike of an option
l	level of an option
v_x	partial derivation of v with respect to x (and analogous)

The standard normal distribution and density functions are defined by

$$n(t) \triangleq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \tag{2}$$

$$\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\, dt \tag{3}$$

$$n_2(x, y; \rho) \triangleq \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) \tag{4}$$

$$\mathcal{N}_2(x, y; \rho) \triangleq \int_{-\infty}^{x}\int_{-\infty}^{y} n_2(u, v; \rho)\, du\, dv \tag{5}$$

4.2. Common Greeks

Delta	Δ	v_x	
Gamma	Γ	v_{xx}	
Theta	Θ	v_t	
Rho	ρ	v_r	in the one-stock model
Rhor	ρ_r	v_r	in the two-stock model
Rhoq	ρ_q	v_q	
Vega	Φ	v_σ	
Kappa	κ	v_ρ	correlation sensitivity (two-stock model)

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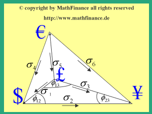

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4.3. Not so common Greeks

Leverage	λ	$\frac{x}{v} v_x$	sometimes Ω, sometimes called "gearing"
Vomma / Volga	Φ'	$v_{\sigma\sigma}$	
Speed		v_{xxx}	
Charm		v_{xt}	
Color		v_{xxt}	
Cross / Vanna		$v_{x\sigma}$	
Forward Delta	Δ^F	v_F	
Driftless Delta	Δ^{dl}	$\Delta e^{q\tau}$	
Dual Theta	DualΘ	v_T	
Strike Delta	Δ^k	v_k	
Strike Gamma	Γ^k	v_{kk}	
Level Delta	Δ^l	v_l	
Level Gamma	Γ^l	v_{ll}	
Beta	β_{12}	$\frac{\sigma_1}{\sigma_2}\rho$	two-stock model

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4.4. Scale-Invariance of Time

Based on the relation

$$v(x_1,\dots,x_n,\tau,r,q_1,\dots,q_n,\sigma_{11},\dots,\sigma_{nn}) = v(x_1,\dots,x_n,\frac{\tau}{a},ar,aq_1,\dots,aq_n,\sqrt{a}\sigma_{11},\dots,\sqrt{a}\sigma_{nn}) \tag{6}$$

we obtain

Theorem 4.1 *(scale invariance of time)*

$$0 = \tau\Theta + r\rho + \sum_{i=1}^{n} q_i \rho_{q_i} + \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{ij}\sigma_{ij}, \tag{7}$$

where Φ_{ij} *denotes the differentiation of* v *with respect to* σ_{ij}*.*

4.5. Scale Invariance of Prices

Definition 4.1 (homogeneity classes) *We call a value function k-homogeneous of degree n if for all $a > 0$*

$$v(ax, ak) = a^n v(x, k). \tag{8}$$

value function strike-homogeneous of degree 1*:* strike-defined option *value function level-homogeneous of degree* 0*:* level-defined option

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4.5.1. Strike-Delta and Strike-Gamma

For a strike-defined value function we have for all $a, b > 0$

$$abv(x,k) = v(abx, abk). \tag{9}$$

We differentiate with respect to a and get for $a = 1$

$$bv(x,k) = bxv_x(bx,bk) + bkv_k(bx,bk). \tag{10}$$

We now differentiate with respect to b get for $b = 1$

$$\begin{aligned} v(x,k) &= xv_x + xv_{xx}x + xv_{xk}k + kv_k + kv_{kx}x + kv_{kk}k & (11)\\ &= x\Delta + x^2\Gamma + 2xkv_{xk} + k\Delta^k + k^2\Gamma^k. & (12)\end{aligned}$$

If we evaluate equation (10) at $b = 1$ we get

$$v = x\Delta + k\Delta^k. \tag{13}$$

We differentiate this equation with respect to k and obtain

$$\begin{aligned} \Delta^k &= xv_{kx} + \Delta^k + k\Gamma^k, & (14)\\ kxv_{kx} &= -k^2\Gamma^k. & (15)\end{aligned}$$

Together with equation (12) we conclude

$$x^2\Gamma = k^2\Gamma^k. \tag{16}$$

4.6. European Options in the Black-Scholes Model

Relations among Greeks for European claims in n-dimensions

$$dS_i(t) = S_i(t)[(r - q_i)\,dt + \sigma_i\,dW_i(t)], \quad i = 1, \ldots, n \tag{17}$$

$$\mathbf{Cov}(W_i(t), W_j(t)) = \rho_{ij} t, \tag{18}$$

where r is the risk-free rate, q_i the dividend rate of asset i or foreign interest rate of exchange rate i, σ_i the volatility of asset i and $(W_1, \ldots, W_n)$ a standard Brownian motion (under the risk-neutral measure) with correlation matrix ρ. Let v denote today's value of the payoff $f(S_1(T), \ldots, S_n(T))$ at maturity T. Then it is known that v satisfies the *Black-Scholes partial differential equation*

$$0 = -v_\tau - rv + \sum_{i=1}^{n} x_i (r - q_i) v_{x_i} + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \circ \sigma^T)_{ij} x_i x_j v_{x_i x_j}. \tag{19}$$

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4.6.1. Relations among Greeks Based on the Log-Normal Distribution

The value function v has a representation given by the n-fold integral

$$v = e^{-r\tau} \int f\left(\ldots, S_i(0)e^{\sigma_i\sqrt{\tau}x_i+\mu_i\tau}, \ldots\right) g(\vec{x}, \rho)\, d\vec{x}, \tag{20}$$

where $\mu_i = r - q_i - \frac{1}{2}\sigma_i^2$ and $g(\vec{x}, \rho)$ is the n-variate standard normal density with correlation matrix ρ. Since we do not want to assume differentiability of the payoff f, but we know that the transition density g is differentiable, we define a change the variables $y_i \stackrel{\Delta}{=} S_i(0)e^{\sigma_i\sqrt{\tau}x_i+\mu_i\tau}$, which leads to

$$v = e^{-r\tau} \int f(\ldots, y_i, \ldots) g\left(\frac{\ln\frac{y_i}{S_i(0)} - \mu_i\tau}{\sigma_i\sqrt{\tau}}, \rho\right) \frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}}. \tag{21}$$

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4.6.2. Properties of the Normal Distribution

We collect some properties of the multivariate normal density function g. We suppose that the vector X of n random variables with means zero and unit variances has a nonsingular normal multivariate distribution with probability density function

$$g(x_1, \ldots, x_n; c_{11}, \ldots, c_{nn}) = (2\pi)^{-\frac{1}{2}n} |\mathbf{C}|^{\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{C}\mathbf{x}\right). \tag{22}$$

Here $\mathbf{C}$ is the inverse of the covariance matrix of X, which is denoted by ρ.

Theorem 4.2 *(Plackett's Identity, 1954) [10]*

$$\frac{\partial g}{\partial \rho_{ij}} = \frac{\partial^2 g}{\partial x_i \partial x_j}. \tag{23}$$

In the two-dimensional case:

$$\frac{\partial n_2(x, y; \rho)}{\partial \rho} = \frac{\partial^2 n_2(x, y; \rho)}{\partial x \partial y}, \tag{24}$$

extends to the corresponding cumulative distribution function, i.e.,

$$\frac{\partial \mathcal{N}_2(x, y; \rho)}{\partial \rho} = \frac{\partial^2 \mathcal{N}_2(x, y; \rho)}{\partial x \partial y} = n_2(x, y; \rho). \tag{25}$$

4.6.3. Correlation Risk and Cross-Gamma

Using the abbreviation $g_{jk} \triangleq \frac{\partial^2 g}{\partial x_j \partial x_k}$ the cross-gamma and correlation risk are

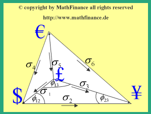
$$\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} = e^{-r\tau}\frac{1}{S_j(0)S_k(0)\sigma_j\sigma_k\tau}\int f(\ldots,y_i,\ldots)g_{jk}\,\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}} \tag{26}$$

$$\frac{\partial v}{\partial \rho_{jk}} = e^{-r\tau}\int f(\ldots,y_i,\ldots)g_{\rho_{jk}}\,\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}}. \tag{27}$$

Invoking Plackett's identity (23) saying that $g_{\rho_{jk}} = g_{jk}$ leads to

Theorem 4.3 *(cross-gamma-correlation-risk relationship)*

$$\frac{\partial v}{\partial \rho_{jk}} = S_j(0)S_k(0)\sigma_j\sigma_k\tau\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)}. \tag{28}$$



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4.6.4. Interest Rate Risk and Delta

A similar computation yields

Theorem 4.4 *(delta-rho relationship)*

$$\frac{\partial v}{\partial q_j} = -S_j(0)\tau \frac{\partial v}{\partial S_j(0)}, \tag{29}$$

$$\frac{\partial v}{\partial r} = -\tau \left(v - \sum_{j=1}^{n} S_j(0) \frac{\partial v}{\partial S_j(0)} \right). \tag{30}$$

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4.6.5. Volatility Risk and Gamma

Theorem 4.5 *(gamma-vega relationship)*

$$\sigma_j \frac{\partial v}{\partial \sigma_j} = \sum_{k=1}^{n} \rho_{jk} \sigma_j \sigma_k S_j(0) S_k(0) \tau \frac{\partial^2 v}{\partial S_j(0) \partial S_k(0)}. \tag{31}$$

4.7. Results for European Claims in the Black-Scholes Model (One-Dimensional Case)

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi \quad \text{scale invariance of time} \tag{32}$$

$$v = x\Delta + k\Delta^k \quad \text{price homogeneity and strikes} \tag{33}$$

$$x^2\Gamma = k^2\Gamma^k \quad \text{price homogeneity and strikes} \tag{34}$$

$$x\Delta = -l\Delta^l \quad \text{price homogeneity and levels} \tag{35}$$

$$x^2\Gamma + x\Delta = l^2\Gamma^l + l\Delta^l \quad \text{price homogeneity and levels} \tag{36}$$

$$\rho = -\tau(v - x\Delta) \quad \text{delta-rho relationship} \tag{37}$$

$$\rho + \rho_q = -\tau v \quad \text{rates symmetry} \tag{38}$$

$$rv = \Theta + (r-q)x\Delta + \frac{1}{2}\sigma^2 x^2\Gamma \quad \text{Black-Scholes PDE} \tag{39}$$

$$qv = \Theta + (q-r)k\Delta^k + \frac{1}{2}\sigma^2 k^2\Gamma^k \quad \text{dual Black-Scholes (strike)} \tag{40}$$

$$rv = \Theta + (q-r+\sigma^2)l\Delta^l + \frac{1}{2}\sigma^2 l^2\Gamma^l \quad \text{dual Black-Scholes (level)} \tag{41}$$

$$\rho_q = -\tau x\Delta \quad \text{delta-rho relationship} \tag{42}$$

$$\rho = -\tau k\Delta^k \quad \text{combination of (42) and (33)} \tag{43}$$

$$\Phi = \sigma\tau x^2\Gamma \quad \text{gamma-vega relationship} \tag{44}$$

4.8. A European Claim in the Two-Dimensional Black-Scholes Model

Relations among the Greeks

$$
\begin{aligned}
0 &= \rho_{q_1} + S_1(0)\tau\Delta_1, && (45)\\
0 &= \rho_{q_2} + S_2(0)\tau\Delta_2, && (46)\\
0 &= q_1\rho_{q_1} + q_2\rho_{q_2} + \frac{1}{2}\sigma_1\Phi_1 + \frac{1}{2}\sigma_2\Phi_2 + r\rho_r + \tau\Theta, && (47)\\
0 &= \Theta - rv + (r-q_1)S_1(0)\Delta_1 + (r-q_2)S_2(0)\Delta_2 \\
&\quad + \frac{1}{2}\sigma_1^2 S_1(0)^2\Gamma_{11} + \rho\sigma_1\sigma_2 S_1(0)S_2(0)\Gamma_{12} + \frac{1}{2}\sigma_2^2 S_2(0)^2\Gamma_{22}, && (48)\\
\kappa &= \sigma_1\sigma_2\tau S_1(0)S_2(0)\Gamma_{12}, && (49)\\
0 &= \rho\kappa - \sigma_1\Phi_1 + \sigma_1^2\tau S_1(0)^2\Gamma_{11}, && (50)\\
0 &= \rho\kappa - \sigma_2\Phi_2 + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, && (51)\\
0 &= \sigma_1\Phi_1 - \sigma_2\Phi_2 - \sigma_1^2\tau S_1(0)^2\Gamma_{11} + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, && (52)\\
\rho_r &= -\tau\left(v - S_1(0)\Delta_1 - S_2(0)\Delta_2\right), && (53)\\
0 &= \tau v + \rho_{q_1} + \rho_{q_2} + \rho_r. && (54)
\end{aligned}
$$

4.9. European Options on the Minimum/Maximum of Two Assets

$$[\phi\,(\eta\min(\eta S_1(T), \eta S_2(T)) - K)]^+ . \tag{55}$$

This is a European put or call on the minimum ($\eta = +1$) or maximum ($\eta = -1$) of the two assets $S_1(T)$ and $S_2(T)$ with strike K. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put. Its value function has been published in Stulz [1982] [14] and can be written as

$$\begin{aligned} & v(t, S_1(t), S_2(t), K, T, q_1, q_2, r, \sigma_1, \sigma_2, \rho, \phi, \eta) \\ &= \phi \Big[S_1(t) e^{-q_1\tau} \mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1) \\ &\quad + S_2(t) e^{-q_2\tau} \mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2) \\ &\quad - K e^{-r\tau} \left(\frac{1-\phi\eta}{2} + \phi \mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho) \right) \Big], \end{aligned} \tag{56}$$

$$\sigma^2 \triangleq \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2, \tag{57}$$

$$\rho_1 \triangleq \frac{\rho\sigma_2 - \sigma_1}{\sigma}, \tag{58}$$

$$\rho_2 \triangleq \frac{\rho\sigma_1 - \sigma_2}{\sigma}, \tag{59}$$

$$d_1 \triangleq \frac{\ln(S_1(t)/K) + (r - q_1 + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \tag{60}$$

$$d_2 \triangleq \frac{\ln(S_2(t)/K) + (r - q_2 + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \tag{61}$$

$$d_3 \triangleq \frac{\ln(S_2(t)/S_1(t)) + (q_1 - q_2 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{62}$$

$$d_4 \triangleq \frac{\ln(S_1(t)/S_2(t)) + (q_2 - q_1 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{63}$$

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4.9.1. Greeks

Delta. Space homogeneity implies that

$$v = S_1(t)\frac{\partial v}{\partial S_1(t)} + S_2(t)\frac{\partial v}{\partial S_2(t)} + K\frac{\partial v}{\partial K}. \tag{64}$$

read off the deltas:

$$\frac{\partial v}{\partial S_1(t)} = \phi e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1), \tag{65}$$

$$\frac{\partial v}{\partial S_2(t)} = \phi e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2), \tag{66}$$

$$\frac{\partial v}{\partial K} = -\phi e^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho)\right). \tag{67}$$

Gamma. We use the identities

$$\frac{\partial}{\partial x}\mathcal{N}_2(x,y;\rho) = n(x)\mathcal{N}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right), \tag{68}$$

$$\frac{\partial}{\partial y}\mathcal{N}_2(x,y;\rho) = n(y)\mathcal{N}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right), \tag{69}$$

and obtain

$$\frac{\partial^2 v}{\partial(S_1(t))^2} = \frac{\phi e^{-q_1\tau}}{S_1(t)\sqrt{\tau}}\left[\frac{\phi}{\sigma_1}n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3-d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right) - \frac{\eta}{\sigma}n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right], \tag{70}$$

$$\frac{\partial^2 v}{\partial(S_2(t))^2} = \frac{\phi e^{-q_2\tau}}{S_2(t)\sqrt{\tau}}\left[\frac{\phi}{\sigma_2}n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4-d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right) - \frac{\eta}{\sigma}n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2-d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right], \tag{71}$$

$$\frac{\partial^2 v}{\partial S_1(t)\partial S_2(t)} = \frac{\phi\eta e^{-q_1\tau}}{S_2(t)\sigma\sqrt{\tau}}n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1-d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right). \tag{72}$$

Kappa. The sensitivity with respect to correlation is directly related to the cross-gamma

$$\frac{\partial v}{\partial \rho} = \sigma_1 \sigma_2 \tau S_1(t) S_2(t) \frac{\partial^2 v}{\partial S_1(t) \partial S_2(t)}. \tag{73}$$

Vega. We refer to (50) and (51) to get the following formulas for the vegas,

$$\begin{aligned}
\frac{\partial v}{\partial \sigma_1} &= \frac{\rho v_\rho + \sigma_1^2 \tau (S_1(t))^2 v_{S_1(t) S_1(t)}}{\sigma_1} && (74) \\
&= S_1(t) e^{-q_1 \tau} \sqrt{\tau} \left[\rho_1 \phi \eta n(d_3) \mathcal{N} \left(\phi \sigma \frac{d_1 - d_3 \rho_1}{\sigma_2 \sqrt{1 - \rho^2}} \right) \right. \\
&\quad \left. + n(d_1) \mathcal{N} \left(\eta \sigma \frac{d_3 - d_1 \rho_1}{\sigma_2 \sqrt{1 - \rho^2}} \right) \right], && (75) \\
\frac{\partial v}{\partial \sigma_2} &= \frac{\rho v_\rho + \sigma_2^2 \tau (S_2(t))^2 v_{S_2(t) S_2(t)}}{\sigma_2} && (76) \\
&= S_2(t) e^{-q_2 \tau} \sqrt{\tau} \left[\rho_2 \phi \eta n(d_4) \mathcal{N} \left(\phi \sigma \frac{d_2 - d_4 \rho_2}{\sigma_1 \sqrt{1 - \rho^2}} \right) \right. \\
&\quad \left. + n(d_2) \mathcal{N} \left(\eta \sigma \frac{d_4 - d_2 \rho_2}{\sigma_1 \sqrt{1 - \rho^2}} \right) \right]. && (77)
\end{aligned}$$

Rho. Looking at (45), (46) and (53) the rhos are given by

$$\frac{\partial v}{\partial q_1} = -S_1(t)\tau \frac{\partial v}{\partial S_1(t)}, \tag{78}$$

$$\frac{\partial v}{\partial q_2} = -S_2(t)\tau \frac{\partial v}{\partial S_2(t)}, \tag{79}$$

$$\frac{\partial v}{\partial r} = -K\tau \frac{\partial v}{\partial K}. \tag{80}$$

Theta. Among the various ways to compute theta one may use the one based on (47).

$$\frac{\partial v}{\partial t} = -\frac{1}{\tau}\left[q_1 v_{q_1} + q_2 v_{q_2} + r v_r + \frac{\sigma_1}{2} v_{\sigma_1} + \frac{\sigma_2}{2} v_{\sigma_2}\right]. \qquad (81)$$

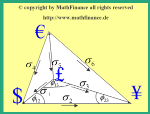
4.10. Heston's Stochastic Volatility Model

$$
\begin{aligned}
dS_t &= S_t \left[\mu \, dt + \sqrt{v(t)} dW_t^{(1)} \right], & (82) \\
dv_t &= \kappa(\theta - v_t) \, dt + \sigma \sqrt{v(t)} dW_t^{(2)}, & (83) \\
\mathbf{Cov} \left[dW_t^{(1)}, dW_t^{(2)} \right] &= \rho \, dt, & (84) \\
\Lambda(S, v, t) &= \lambda v. & (85)
\end{aligned}
$$

Heston provides a closed-form solution for European vanilla options paying

$$
\left[\phi \left(S_T - K \right) \right]^+ . \tag{86}
$$

As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put, K the strike in units of the domestic currency



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4.10.1. Abbreviations

$$a \triangleq \kappa\theta \tag{87}$$

$$u_1 \triangleq \frac{1}{2} \tag{88}$$

$$u_2 \triangleq -\frac{1}{2} \tag{89}$$

$$b_1 \triangleq \kappa + \lambda - \sigma\rho \tag{90}$$

$$b_2 \triangleq \kappa + \lambda \tag{91}$$

$$d_j \triangleq \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)} \tag{92}$$

$$g_j \triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{b_j - \rho\sigma\varphi i - d_j} \tag{93}$$

$$\tau \triangleq T - t \tag{94}$$

$$D_j(\tau,\varphi) \triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right] \tag{95}$$

$$\begin{aligned} C_j(\tau,\varphi) \triangleq\ & (r-q)\varphi i\tau \\ & + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\varphi i + d)\tau - 2\ln\left[\frac{1 - g_j e^{d_j\tau}}{1 - e^{d_j\tau}}\right]\right\} \end{aligned} \tag{96}$$

(97)

$$f_j(x,v,t,\varphi) \triangleq e^{C_j(\tau,\varphi)+D_j(\tau,\varphi)v+i\varphi x} \tag{98}$$

$$P_j(x,v,\tau,y) \triangleq \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\varphi y} f_j(x,v,\tau,\varphi)}{i\varphi}\right] d\varphi \tag{99}$$

$$p_j(x,v,\tau,y) \triangleq \frac{1}{\pi}\int_0^\infty \Re\left[e^{-i\varphi y} f_j(x,v,\tau,\varphi)\right] d\varphi \tag{100}$$

$$P_+(\phi) \triangleq \frac{1-\phi}{2}+\phi P_1(\ln S_t, v_t, \tau, \ln K) \tag{101}$$

$$P_-(\phi) \triangleq \frac{1-\phi}{2}+\phi P_2(\ln S_t, v_t, \tau, \ln K) \tag{102}$$

This notation is motivated by the fact that the numbers P_j are the cumulative distribution functions (in the variable y) of the log-spot price after time τ starting at x for some drift μ. The numbers p_j are the respective densities.

4.10.2. Value

The value function for European vanilla options is given by

$$V = \phi \left[e^{-q\tau} S_t P_+(\phi) - K e^{-r\tau} P_-(\phi) \right] \tag{103}$$

The value function takes the form of the Black-Scholes formula for vanilla options. The probabilities $P_\pm(\phi)$ correspond to $\mathcal{N}(\phi d_\pm)$ in the constant volatility case.

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4.10.3. Greeks

Spot delta.

$$\Delta \triangleq \frac{\partial V}{\partial S_t} = \phi e^{-q\tau} P_+(\phi) \tag{104}$$

Dual delta.

$$\Delta^K \triangleq \frac{\partial V}{\partial K} = -\phi e^{-r\tau} P_-(\phi) \tag{105}$$

Gamma.

$$\Gamma \triangleq \frac{\partial \Delta}{\partial S_t} = \frac{\partial \Delta}{\partial x}\frac{\partial x}{\partial S_t} = \frac{e^{-q\tau}}{S_t} p_1(\ln S_t, v_t, \tau, \ln K) \tag{106}$$

Dual Gamma.

$$\Gamma^K \triangleq \frac{\partial \Delta^K}{\partial K} = \frac{\partial \Delta^K}{\partial y}\frac{\partial y}{\partial K} = \frac{e^{-r\tau}}{K} p_1(\ln S_t, v_t, \tau, \ln K) \tag{107}$$

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Rho. Rho is connected to delta via equations (43) and (42).

$$\frac{\partial V}{\partial r} = \phi K e^{-r\tau} \tau P_{-}(\phi), \tag{108}$$

$$\frac{\partial V}{\partial q} = -\phi S_t e^{-q\tau} \tau P_{+}(\phi). \tag{109}$$

Theta. Theta can be computed using the partial differential equation for the Heston vanilla option

$$V_t + (r-q)SV_S + \frac{1}{2}\sigma v V_{vv} + \frac{1}{2} v S^2 V_{SS} + \rho\sigma v S V_{vS} - qV + [\kappa(\theta - v) - \lambda]V_v = 0, \tag{110}$$

where the derivatives with respect to initial variance v must be evaluated numerically.

4.11. Summary

- Understand homogeneity-based methods to compute analytical formulas of Greeks for analytically known value functions of options in a one-and higher-dimensional market
- Restricting the view to the Black-Scholes model there are numerous further relations between various Greeks
- Saving computation time for the mathematician who has to differentiate complicated formulas as well as for the computer, because analytical results for Greeks are usually faster to evaluate than finite differences involving at least twice the computation of the option's value
- Knowing how the Greeks are related among each other can speed up finite-difference-, tree-, or Monte Carlo-based computation of Greeks or lead at least to a quality check
- Many of the results are valid beyond the Black-Scholes model
- Most remarkably some relations of the Greeks are based on properties of the normal distribution refreshing the active interplay between mathematics and financial markets.

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