Efficient Computation of Option Price Sensitivities Using Homogeneity and other Tricks

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Abstract

No front-office software can survive without providing derivatives of options prices with respect to underlying market or model parameters, the so called Greeks. We present a list of common Greeks and exploit homogeneity properties of financial markets to derive relationships between Greeks out of which many are model-independent. We apply our results to European style options, rainbow options, path-dependent options as well as options priced in Heston's stochastic volatility model and show shortcuts to avoid exorbitant and time-consuming computations of derivatives which even strong symbolic calculators fail to produce.

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1 Introduction

Based on homogeneity of time and price level of a financial product we can derive relations for the options sensitivities, the so-called "Greeks". The basic market model we use is the Black-Scholes model with stocks paying a continuous dividend yield and a riskless cash bond. This model supports the homogeneity properties which are valid in general, but its structure is so simple, that we can concentrate on the essential statements of this paper. We will also discuss how to extend out work to more general market models.

We list the commonly used Greeks and their notations. We do not claim this list to be complete, because one can always define more derivatives of the option price function.

First of all we analyze the homogeneity of a financial market, consisting of one stock and one cash bond. The techniques we present are easily expanded to the higher dimensional case as we show later on.

In addition we look at the Greeks of European options in the Black-Scholes model. Essentially it turns out, that one only needs to know two Greeks in order to calculate all the other Greeks without differentiating.

Another interesting example is a European derivative security depending on two assets. For such rainbow options the analysis of the risk due to changing correlation of the two assets is very important. We will show how this risk is related to simultaneous changes of the two underlying securities.

There are several applications of these homogeneity relations.

1. It helps saving time in computing derivatives.

2. It produces a robust implementation compared to Greeks via difference quotients.

3. It allows to check the quality and consistency of Greeks produced by finite-difference-, tree- or Monte Carlo methods.

4. It admits a computation of Greeks for Monte Carlo based values.

5. It shows relationships between Greeks which wouldn't be noticed merely by looking at difference quotients.

1.1 Notation

S	stock price or stock price process
B	cash bond, usually with risk free interest rate r
r	risk free interest rate
q	dividend yield (continuously paid)
σ	volatility of one stock, or volatility matrix of several stocks
ρ	correlation in the two-asset market model
t	date of evaluation ("today")
T	date of maturity
$\tau = T - t$	maturity of an option
x	stock price at time t
$f(\cdot)$	payoff function
$v(x, t, \ldots)$	value of an option
k	strike of an option
l	level of an option
v_x	partial derivation of v with respect to x (and analogous)

The standard normal distribution and density functions are defined by

$$n(t) \triangleq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \tag{1}$$

$$\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\,dt \tag{2}$$

$$n_2(x, y; \rho) \triangleq \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) \tag{3}$$

$$\mathcal{N}_2(x, y; \rho) \triangleq \int_{-\infty}^{x} \int_{-\infty}^{y} n_2(u, v; \rho)\,du\,dv \tag{4}$$

See http://www.MathFinance.de/frontoffice.html for a source code to compute $\mathcal{N}_2$.

1.2 The Greeks

Delta	Δ	v_x	
Gamma	,	v_{xx}	
Theta	Θ	v_t	
Rho	ρ	v_r	in the one-stock model
Rhor	ρ_r	v_r	in the two-stock model
Rhoq	ρ_q	v_q	
Vega	Φ	v_σ	
Kappa	κ	v_ρ	correlation sensitivity (two-stock model)

Greeks, not so commonly used:

Leverage	λ	$\frac{x}{v} v_x$	sometimes Ω, sometimes called "gearing"
Vomma	Φ'	$v_{\sigma\sigma}$	
Speed		v_{xxx}	
Charm		$v_{x\tau}$	
Color		$v_{xx\tau}$	
Forward Delta	Δ^F	v_F	
Driftless Delta	Δ^{dl}	$\Delta e^{q\tau}$	
Dual Theta	DualΘ	v_T	
Strike Delta	Δ^k	v_k	
Strike Gamma	$,^k$	v_{kk}	
Level Delta	Δ^l	v_l	
Level Gamma	$,^l$	v_{ll}	
Beta	β_{12}	$\frac{\sigma_1}{\sigma_2}\rho$	two-stock model

2 Fundamental Properties

2.1 Homogeneity of Time

In most cases the price of the option is not a function of both the current time t **and** the maturity time T, but rather **only** a function of the time to maturity $\tau = T - t$ implying the relations

$$\Theta = v_t = -v_\tau = -v_T = -Dual\Theta \tag{5}$$

This relationship extends naturally to the situation of options depending on several intermediate times such as compound or Bermuda options.

2.2 Scale-Invariance of Time

We may want to measure time in units other than years in which case interest rates and volatilities, which are normally quoted on an annual basis, must be changed according to the following rules for all $a > 0$.

$$\tau \quad \rightarrow \quad \frac{\tau}{a}$$

$$\begin{aligned} r &\to ar \\ q &\to aq \\ \sigma &\to \sqrt{a}\sigma \end{aligned} \tag{6}$$

The option's value must be invariant under this rescaling, i.e.,

$$v(x, \tau, r, q, \sigma, \ldots) = v(x, \frac{\tau}{a}, ar, aq, \sqrt{a}\sigma, \ldots) \tag{7}$$

We differentiate this equation with respect to a and obtain for $a = 1$

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi, \tag{8}$$

a general relation between the Greeks *theta, rho, rhoq* and *vega*. It extends naturally to the case of multiple assets and multiple intermediate dates.

2.3 Scale Invariance of Prices

The general idea is that value of securities may be measured in a different unit, just like values of European stocks are now measured in Euro instead of in-currencies. Option contracts usually depend on strikes and barrier levels. Rescaling can have different effects on the value of the option. Essentially we may consider the following types of homogeneity classes. Let $v(x, k)$ be the value function of an option, where x is the spot (or a vector of spots) and k the strike or barrier or a vector of strikes or barriers. Let a be a positive real number.

Definition 1 (homogeneity classes) *We call a value function k-homogeneous of degree n if for all a*

$$v(ax, ak) = a^n v(x, k). \tag{9}$$

The value function of a European call or put option with strike K is then K-homogeneous of degree 1, a power call with strike K and cap C is both K-homogeneous of degree 2 and C-homogeneous of degree 0, a double barrier call with strike K and barriers $B = (L, H)$ is K-homogeneous of degree 1 and B-homogeneous of degree 0. We will call an option strike-defined, if there is just one strike k and the value function is k-homogeneous of degree 1 and level-defined if there is just one level l and the value function is l-homogeneous of degree 0, e.g. a digital cash call.

2.3.1 Strike-Delta and Strike-Gamma

For a strike-defined value function we have for all $a, b > 0$

$$abv(x, k) = v(abx, abk). \tag{10}$$

We differentiate with respect to a and get for $a = 1$

$$bv(x,k) \quad = \quad bxv_x(bx,bk) + bkv_k(bx,bk). \tag{11}$$

We now differentiate with respect to b get for $b = 1$

$$\begin{aligned} v(x,k) \quad &= \quad xv_x + xv_{xx}x + xv_{xk}k + kv_k + kv_{kx}x + kv_{kk}k && (12)\\ &= \quad x\Delta + x^2, \; + 2xkv_{xk} + k\Delta^k + k^2, {}^k. && (13)\end{aligned}$$

If we evaluate equation (11) at $b = 1$ we get

$$v = x\Delta + k\Delta^k. \tag{14}$$

We differentiate this equation with respect to k and obtain

$$\begin{aligned} \Delta^k \quad &= \quad xv_{kx} + \Delta^k + k, {}^k, && (15)\\ kxv_{kx} \quad &= \quad -k^2, {}^k. && (16)\end{aligned}$$

Together with equation (13) we conclude

$$x^2, \; = k^2, {}^k. \tag{17}$$

2.3.2 Level-Delta and Level-Gamma

For a level-defined value function we have for all $a, b > 0$

$$v(x,l) \quad = \quad v(abx, abl). \tag{18}$$

We differentiate with respect to a and get at $a = 1$

$$0 \quad = \quad v_x(bx,bl)bx + v_l(bx,bl)bl. \tag{19}$$

If we set $b = 1$ we get the relation

$$\Delta x + \Delta^l l = 0. \tag{20}$$

Now we differentiate equation (19) with respect to b and get at $b = 1$

$$0 \quad = \quad v_{xx}x^2 + 2v_{xl}xl + v_{ll}l^2. \tag{21}$$

One the other hand we can differentiate the relation between delta and level-delta with respect to l and get

$$v_{xl}x + l, {}^l + \Delta^l \quad = \quad 0. \tag{22}$$

Together with equation (21) we conclude

$$x^2, \; + x\Delta = l^2, {}^l + l\Delta^l. \tag{23}$$

3 European Options in the Black-Scholes Model

In this section we analyze the European claim in the Black-Scholes model

$$dS_t = S_t[(r-q)dt + \sigma dW_t]. \tag{24}$$

This situation has a simple structure which allows us to obtain even more relations for the Greeks. We assume that the stock pays a continuous dividend yield with rate q. It can also be seen as a foreign exchange rate and q denotes the risk free interest rate in the foreign currency.

3.1 Scale Invariance of Prices

A European option is described by its payoff function and the price of this option in the Black-Scholes model is given by

$$v(x,\tau) \quad = \quad e^{-r\tau}\mathbf{E}^*[f(S(\tau)) \mid S(0)=x]. \tag{25}$$

The expectation is taken under the risk-neutral measure P^*. Now we define a set of European options related to this option by a payoff $f(\frac{x}{a})$ parameterized by a positive real number a. We denote their value function by

$$u(x,\tau,a) \quad = \quad e^{-r\tau}\mathbf{E}^*\left[f\left(\frac{S(\tau)}{a}\right) \,\middle|\, S(0)=x\right]. \tag{26}$$

We now differentiate the equation $u(\frac{x}{a},\tau,1) = u(x,\tau,a)$ with respect to a and get

$$\frac{-1}{a^2}xu_x(\frac{x}{a},\tau,1) \quad = \quad e^{-r\tau}\mathbf{E}^*\left[f'\left(\frac{S(\tau)}{a}\right)\frac{-S(\tau)}{a^2} \,\middle|\, S(0)=x\right]. \tag{27}$$

If we set $a=1$ and use $v(x,\tau) = u(x,\tau,1)$, we obtain the relation

$$x\Delta = \mathbf{E}^*[f'(S(\tau))S(\tau)e^{-r\tau} \mid S(0)=x] \tag{28}$$

Remark 1 *In this calculation we used, that the payoff function is differentiable. This condition can be relaxed by the assumption, that there exists a sequence of differentiable functions f_n, which converges P^* almost surely pointwise to the function f. For instance, f can have a finite number of jumps and be continuous between them. This argumentation is always valid, whenever we differentiate the payoff function under the expectation.*

3.1.1 Interest Rate Risk

We differentiate (25) with respect to r and obtain:

$$\begin{aligned} v_r &= (-\tau)e^{-r\tau}\mathbf{E}^*[f(S(\tau)) \mid X(0)=x] \\ &\quad + e^{-r\tau}\mathbf{E}^*[f'(S(\tau))S(\tau)\tau \mid S(0)=x] && (29) \\ &= -\tau v + \tau\mathbf{E}^*[f'(S(\tau))S(\tau)e^{-r\tau} \mid S(0)=x] && (30) \end{aligned}$$

Comparing this with equation (28) we find the following relation for the Greeks

$$\rho = -\tau(v - x\Delta). \tag{31}$$

This result is not surprising, since $v - x\Delta$ is the amount of money one has to invest in the cash bond if one hedges the option by delta hedge. On the other hand the interest rate risk of a zero-coupon bond is (minus) the amount of money invested times the duration.

3.1.2 Rates Symmetry

Differentiating equation (25) with respect to the dividend yield q leads to

$$v_q \quad = \quad e^{-r\tau}\mathbf{E}^*[f'(S(\tau))S(\tau)(-\tau)|S(0) = x], \tag{32}$$

and a comparison of this with equation (30) to homogeneity relation:

$$\rho + \rho_q = -\tau v. \tag{33}$$

3.2 Black-Scholes PDE

Lemma 1 *For all European contingent claims the following equation (Black-Scholes PDE) holds.*

$$-v_\tau - rv + (r-q)xv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0 \tag{34}$$

Remark 2 *From this lemma we obtain the following relation for the Greeks.*

$$rv = \Theta + (r-q)x\Delta + \frac{1}{2}x^2\sigma^2, \, . \tag{35}$$

3.2.1 Dual Black-Scholes PDEs

One can combine the general results for the dual Greeks with the Black-Scholes PDE and obtains the **dual Black-Scholes equations**:

$$qv \quad = \quad \Theta + (q-r)k\Delta^k + \frac{1}{2}\sigma^2 k^2, {}^k \tag{36}$$

$$rv \quad = \quad \Theta + (q-r+\sigma^2)l\Delta^l + \frac{1}{2}\sigma^2 l^2, {}^l \tag{37}$$

3.3 Results for European Claims in the Black-Scholes Model

We found a lot of relations for European options. Of course, the general relations hold and additionally we proved some more relations for European options in the

Black-Scholes model. We list all these relations now.

$$
\begin{aligned}
0 &= \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi && \text{scale invariance of time} && (38)\\
v &= x\Delta + k\Delta^k && \text{log-price homogeneity and strikes} && (39)\\
x^2, &= k^2,\ ^k && \text{log-price homogeneity and strikes} && (40)\\
x\Delta &= -l\Delta^l && \text{log-price homogeneity and levels} && (41)\\
x^2,\ + x\Delta &= l^2,\ ^l + l\Delta^l && \text{log-price homogeneity and levels} && (42)\\
\rho &= -\tau(v - x\Delta) && \text{Interest rate risk} && (43)\\
\rho + \rho_q &= -\tau v && \text{rates symmetry} && (44)\\
rv &= \Theta + (r-q)x\Delta + \frac{1}{2}\sigma^2 x^2, && \text{Black-Scholes PDE} && (45)\\
qv &= \Theta + (q-r)k\Delta^k + \frac{1}{2}\sigma^2 k^2,\ ^k && \text{Dual Black-Scholes} && (46)\\
rv &= \Theta + (q-r+\sigma^2)l\Delta^l + \frac{1}{2}\sigma^2 l^2,\ ^l && \text{Dual Black-Scholes} && (47)
\end{aligned}
$$

Lemma 2 *From the relations above we conclude*

$$
\begin{aligned}
\rho &= -\tau k\Delta^k, && (48)\\
\rho_q &= -\tau x\Delta. && (49)\\
\Phi &= \sigma\tau x^2,\ , && (50)
\end{aligned}
$$

Proof. Equation (48) follows from (39) and (43). From (43) and (44) one easily obtains (49). The proof of (50) starts with (38) and equation (45) times τ. Eliminating $\tau\Theta$ yields

$$r\rho + q\rho_q + \frac{1}{2}\sigma\Phi = -rv\tau + (r-q)\tau x\Delta + \frac{1}{2}\sigma^2\tau x^2, \tag{51}$$

Using (43) to get rid of $x\Delta$ we get:

$$q\rho_q + \frac{1}{2}\sigma\Phi \quad = \quad -q\rho - qv\tau + \frac{1}{2}\sigma^2\tau x^2, \tag{52}$$

All terms containing q vanish because of (44) which establishes equation (50). $\square$
An interpretation of equation (50) can be found in [6]. We would like to point out that this relationship is based on a fact concerning the normal distribution function, namely defining

$$
\begin{aligned}
n(t,\sigma) &\triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}, && (53)\\
\mathcal{N}(x,\sigma) &\triangleq \int_{-\infty}^{x} n(t,\sigma)\,dt, && (54)
\end{aligned}
$$

one can verify that

$$\sigma\partial^2_{xx}\mathcal{N}(x,\sigma) = \partial_\sigma\mathcal{N}(x,\sigma). \tag{55}$$

There are surely more relations one can prove, but the next theorem will give a deeper insight into the relations of the Greeks.

Theorem 1 *If the price and two Greeks g_1, g_2 of a European option are given with*

$$g_1 \in G_1 = \{\Delta, \Delta^k, \Delta^l, \rho, \rho_q\}, \tag{56}$$
$$g_2 \in G_2 = \{, , , ^k, , , ^l, \Phi, \Theta\}, \tag{57}$$

then all the other Greeks ($\in G_1 \cup G_2$) can be calculated. Furthermore, if Θ and another Greek from G_2 is given, it is also possible, to determine all other Greeks.

Proof. The relations (38) to (45) are independent of each other. The relations (46) to (49) are conclusions. To get a overview over all these relations, we list the appearance of each Greek in all these relations. With X or O we denote, that the marked Greek appears in the relation. The relations marked with X show, that there is a relation between Greeks of G_1 and G_2 and the O shows, that this relation concerns only the Greeks of one set.

		Greeks $\in G_1$					*Greeks* $\in G_2$				
equation	v	Δ	Δ^k	Δ^l	ρ	ρ_q	,	$,^k$	$,^l$	Φ	Θ
(38)					X	X				X	X
(39)	O	O	O								
(40)							O	O			
(41)		O		O							
(42)		X		X			X		X		
(43)	O	O			O						
(44)	O				O	O					
(45)	X	X					X				X
(46)	X		X					X			X
(47)	X			X					X		X
(48)			O		O						
(50)							O			O	
(49)		O				O					

Let us now assume the option price and one Greek from the set G_1 are given. Then a look at the table shows that all Greeks of the set G_1 can be evaluated. If all Greeks of the set G_1 are known and additionally one Greek of the set G_2 is given, all other Greeks can be determined. One the other hand, only eight equations are independent, so the knowledge of two Greeks is also the minimum knowledge one needs to determine all ten Greeks. This is the proof of the first statement.

If Θ and another Greek from G_2 is given, then it is always possible to determine one Greek of the set G_1 and one applies the part of this theorem already proved. If , , ^{k} or , ^{l} is given, one can use one of the Black-Scholes equations (45) to (47). If vega Φ is given, one can use (50) to get , . □

4 Examples for the One-Dimensional Case

4.1 Vanilla Calls and Puts in a Foreign Exchange Market

In the special case of plain vanilla calls and puts in a foreign exchange market all relations for the Greeks presented above are valid. These formulas are well known and can be found in [7].

4.2 Compound Options

A *compound option* is an option on an option, i.e., a call on a call or a call on a put or a put on a call or a put on a put. For comparison you may consult [1].

4.2.1 abbreviations

- K: strike of the underlying option
- k: strike of the compound option
- ϕ: +1 (-1) is the underlying option is a call (put)
- ω: +1 (-1) is the compound option is a call (put)
- $\theta_\pm \stackrel{\Delta}{=} \frac{r-q}{\sigma} \pm \frac{\sigma}{2}$
- $S_t = S_0 e^{\sigma W_t + \sigma\theta_- t}$: price of the underlying at time t
- $d_\pm^\tau \stackrel{\Delta}{=} \frac{\ln\frac{S_t}{K} + \sigma\theta_\pm\tau}{\sigma\sqrt{\tau}}$
- $d_\pm \stackrel{\Delta}{=} \frac{\ln\frac{S_0}{K} + \sigma\theta_\pm T}{\sigma\sqrt{T}}$
- $\text{Vanilla}(S_t, K, \sigma, r, q, \tau, \phi) = \phi(S_t e^{-q\tau}\mathcal{N}(\phi d_+^\tau) - K e^{-r\tau}\mathcal{N}(\phi d_-^\tau))$
- $H(x) \stackrel{\Delta}{=} \text{Vanilla}(S_0 e^{\sigma\sqrt{t}x + \sigma\theta_- t}, K, \sigma, r, q, \tau, \phi) - k$
- X: the unique number satisfying $H(X) = 0$
- $e \stackrel{\Delta}{=} \frac{X\sqrt{T} + \sqrt{t}d_-}{\sqrt{\tau}}$

The value of a compound option is given by

$$
\begin{aligned}
& v(S_0, K, k, T, t, \sigma, r, q, \phi, \omega) \qquad (58)\\
= \quad & e^{-rt}\int_{y=-\infty}^{y=+\infty} \left[\omega\left(\text{Vanilla}\left(S_0 e^{\sigma\sqrt{t}y + \sigma\theta_- t}, K, \sigma, r, q, \tau, \phi\right) - k\right)\right]^+ n(y)\,dy \\
= \quad & \phi\omega S_0 e^{-qT}\mathcal{N}_2\left(-\phi\omega(X - \sigma\sqrt{t}), \phi d_+; \omega\sqrt{\frac{t}{T}}\right) \\
- \quad & \phi\omega K e^{-rT}\mathcal{N}_2\left(-\phi\omega X, \phi d_-; \omega\sqrt{\frac{t}{T}}\right) \\
- \quad & \omega k e^{-rt}\mathcal{N}(-\phi\omega X).
\end{aligned}
$$

Delta The value function v is (K,k)-homogeneous of degree 1, whence v has the representation

$$v = S_0 \frac{\partial v}{\partial S_0} + K \frac{\partial v}{\partial K} + k \frac{\partial v}{\partial k}, \tag{59}$$

and the deltas can be just read off, e.g.,

$$\frac{\partial v}{\partial S_0} = \phi\omega e^{-qT} \mathcal{N}_2\left(-\phi\omega(X - \sigma\sqrt{t}), \phi d_+; \omega\sqrt{\frac{t}{T}}\right). \tag{60}$$

Gamma. To compute gamma, we need to know the derivative of X, which is actually a function $X = X(S_0, K, k, T, t, \sigma, r, q, \phi)$. Although we can not determine this function explicitly, we can compute its derivatives explicitly using implicit differentiation. We obtain

$$\frac{\partial X}{\partial S_0} = -\frac{\frac{\partial H}{\partial S_0}}{\frac{\partial H}{\partial x}} = -\frac{\phi e^{-q\tau}\mathcal{N}(\phi d_+^\tau) e^{\sigma\sqrt{t}X + \sigma\theta_- t}}{\phi e^{-q\tau}\mathcal{N}(\phi d_+^\tau) S_0 e^{\sigma\sqrt{t}X + \sigma\theta_- t}\sigma\sqrt{t}} = \frac{-1}{S_0\sigma\sqrt{t}}, \tag{61}$$

and hence for gamma

$$\frac{\partial^2 v}{\partial S_0^2} = \frac{e^{-qT}}{\sigma S_0}\left[\frac{1}{\sqrt{t}} n(X - \sigma\sqrt{t})\mathcal{N}(\phi d_+^\tau) + \frac{\omega}{\sqrt{T}} n(d_+)\mathcal{N}(-\phi\omega e)\right]. \tag{62}$$

Theta. To find theta we use the Black-Scholes PDE and get

$$\begin{aligned}
\Theta &= \phi\omega q S_0 e^{-qT}\mathcal{N}_2\left(-\phi\omega(X - \sigma\sqrt{t}), \phi d_+; \omega\sqrt{\frac{t}{T}}\right) \qquad (63)\\
&- \phi\omega r K e^{-rT}\mathcal{N}_2\left(-\phi\omega X, \phi d_-; \omega\sqrt{\frac{t}{T}}\right)\\
&- \omega r k e^{-rt}\mathcal{N}(-\phi\omega X)\\
&- \frac{\sigma}{2} S_0 e^{-qT}\left[\frac{1}{\sqrt{t}} n(X - \sigma\sqrt{t})\mathcal{N}(\phi d_+^\tau) + \frac{\omega}{\sqrt{T}} n(d_+)\mathcal{N}(-\phi\omega e)\right].
\end{aligned}$$

Intuitively, we deduce a formula for v_τ by just omitting the terms involving t and obtain

$$\begin{aligned}
-\frac{\partial v}{\partial \tau} &= \phi\omega q S_0 e^{-qT}\mathcal{N}_2\left(-\phi\omega(X - \sigma\sqrt{t}), \phi d_+; \omega\sqrt{\frac{t}{T}}\right) \qquad (64)\\
&- \phi\omega r K e^{-rT}\mathcal{N}_2\left(-\phi\omega X, \phi d_-; \omega\sqrt{\frac{t}{T}}\right)\\
&- \frac{\sigma}{2} S_0 e^{-qT}\frac{\omega}{\sqrt{T}} n(d_+)\mathcal{N}(-\phi\omega e).
\end{aligned}$$

We now extend the scale invariance of time equation (38) to

$$\Theta = \frac{\tau}{t} v_\tau - \frac{r}{t} v_r - \frac{q}{t} v_q - \frac{\sigma}{2t} v_\sigma, \tag{65}$$

compare this formula with (63) and can instantly read off vega and rho.

Vega.

$$\frac{\partial v}{\partial \sigma} = S_0 e^{-qT}\left[\sqrt{t}n(X-\sigma\sqrt{t})\mathcal{N}(\phi d_+^\tau) + \omega\sqrt{T}n(d_+)\mathcal{N}(-\phi\omega e)\right]. \quad (66)$$

Rho.

$$\frac{\partial v}{\partial r} = \phi\omega T K e^{-rT}\mathcal{N}_2\left(-\phi\omega X, \phi d_-; \omega\sqrt{\frac{t}{T}}\right) + \omega t k e^{-rt}\mathcal{N}(-\phi\omega X) \quad (67)$$

$$\frac{\partial v}{\partial q} = -\phi\omega T S_0 e^{-qT}\mathcal{N}_2\left(-\phi\omega(X-\sigma\sqrt{t}), \phi d_+; \omega\sqrt{\frac{t}{T}}\right). \quad (68)$$

5 A European Claim in the Two-dimensional Black-Scholes Model

5.1 Pricing of a European Option

Rainbow options are financial instruments which depend on several risky assets. Many of them are very sensitive to changes of correlation. we call kappa (κ) the derivative of the option value v with respect to the correlation ρ.
The computational effort to compute the kappa is hard, even in a simple framework, but in the Black-Scholes model with two stocks and one cash bond we find a cross-gamma-correlation-risk relationship which can be used easily to find kappa.
Let the stock price processes S_1 and S_2 be described by

$$\ln\frac{S_1(\tau)}{S_1(0)} = (r - q^1 - \frac{1}{2}\sigma_1^2)\tau + \sigma_1 W_\tau^1, \quad (69)$$

$$\ln\frac{S_2(\tau)}{S_2(0)} = (r - q^2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2\rho W_\tau^1 + \sigma_2\sqrt{1-\rho^2}W_\tau^2. \quad (70)$$

W^1 and W^2 are two independent Brownian motions under the risk neutral measure. The probability density for the distribution of $S^1(\tau)$ is denoted by $h_1(x)$ and is given by the log normal density:

$$h_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2\tau}}\frac{1}{x}\exp\left(-\frac{A^2}{2\sigma_1^2\tau}\right), \quad (71)$$

$$A \triangleq \ln\left(\frac{x}{S_1(0)}\right) - r\tau + q^1\tau + \frac{1}{2}\sigma_1^2\tau. \quad (72)$$

The equation for the second stock price process can be written as

$$\ln\frac{S_2(\tau)}{S_2(0)} = (r - q^2 - \frac{1}{2}\sigma_2^2)\tau + \frac{\sigma_2\rho}{\sigma_1}\left(\ln\left(\frac{S_1(\tau)}{S_1(0)}\right) - (r - q^1 - \frac{1}{2}\sigma_1^2)\tau\right) + \sigma_2\sqrt{1-\rho^2}W_\tau^2. \quad (73)$$

The conditional distribution of $S_2(\tau)$ given $S_1(\tau)$ is thus log-normal with density

$$
\begin{aligned}
h_2(S_2|S_1(\tau)=x)(y) &= \frac{1}{y\sqrt{2\pi\sigma_2^2(1-\rho^2)\tau}} \exp\left(-\frac{B^2}{2\sigma_2^2(1-\rho^2)\tau}\right), && (74)\\
B &\triangleq \left[\ln\left(\frac{y}{S_2(0)}\right) - r\tau + q^2\tau + \frac{1}{2}\sigma_2^2\tau - \frac{\sigma_2\rho}{\sigma_1}A\right]. && (75)
\end{aligned}
$$

The joint distribution of $S_1(\tau)$ and $S_2(\tau)$ is given by the product of h_1 and h_2

$$h(x,y) = h_1(x)\cdot h_2(S_2(\tau) \mid S_1(\tau)=x)(y). \tag{76}$$

A European option with maturity τ and payoff $f(S_1(\tau), S_2(\tau))$ will be priced by

$$v = e^{-r\tau}\int_0^\infty\int_0^\infty h(x,y)\cdot f(x,y)\,dx\,dy. \tag{77}$$

5.2 Calculation of the Greeks

From the general pricing of a European claim in a two-dimensional Black-Scholes market we can compute all the derivatives of the option price. We suppress the arguments of the functions and we assume, that we are allowed to differentiate under the integral. Therefore, we only have to differentiate the function $\tilde{h}(x,y) \triangleq e^{-r\tau}h(x,y)$ with respect to the parameters resulting into the equations

$$
\begin{aligned}
\partial_\rho \tilde{h} &= \frac{\rho}{1-\rho^2}E_1 - \frac{\sigma_2}{\sigma_1}E_4 + \frac{2\rho}{1-\rho^2}E_6, && (78)\\
\partial_{S_1(0)} \tilde{h} &= \frac{-1}{S_1(0)}E_2 + \frac{\sigma_2\rho}{\sigma_1 S_1(0)}E_5, && (79)\\
\partial_{S_2(0)} \tilde{h} &= \frac{-1}{S_2(0)}E_5, && (80)\\
\partial_{\sigma_1} \tilde{h} &= \frac{-1}{\sigma_1}E_1 - \frac{2}{\sigma_1}E_3 + \sigma_1\tau E_2 + \frac{\sigma_2\rho}{\sigma_1^2}E_4 - \sigma_2\rho\tau E_5, && (81)\\
\partial_{\sigma_2} \tilde{h} &= \frac{-1}{\sigma_2}E_1 - \frac{2}{\sigma_2}E_6 + \sigma_2\tau E_5 - \frac{\rho}{\sigma_1}E_4, && (82)\\
\partial_{q^1} \tilde{h} &= \tau E_2 - \frac{\sigma_2\rho\tau}{\sigma_1}E_5, && (83)\\
\partial_{q^2} \tilde{h} &= \tau E_5, && (84)\\
\partial_r \tilde{h} &= -\tau E_1 - \tau E_2 + \left(\frac{\sigma_2\rho\tau}{\sigma_1} - \tau\right)E_5, && (85)\\
\partial_\tau \tilde{h} &= -(r+\frac{1}{\tau})E_1 - \frac{1}{\tau}E_3 + (-r+q^1+\frac{1}{2}\sigma_1^2)E_2 - \frac{1}{\tau}E_6\\
&\quad + \left(-r+q^2+\frac{1}{2}\sigma_2^2 - \frac{\sigma_2\rho}{\sigma_1}(-r+q^1+\frac{1}{2}\sigma_1^2)\right)E_5, && (86)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{S_1(0)}\partial_{S_1(0)}\tilde{h} &= \frac{-1}{\sigma_1^2(1-\rho^2)\tau S_1(0)^2}E_1 + \frac{1}{S_1(0)^2}E_2 - \frac{2}{\sigma_1^2 \tau S_1(0)^2}E_3 \\
&\quad + \frac{2\sigma_2\rho}{\sigma_1^3\tau S_1(0)^2}E_4 - \frac{\sigma_2\rho}{\sigma_1 S_1(0)^2}E_5 \\
&\quad - \frac{2\rho^2}{\sigma_1^2(1-\rho^2)\tau S_1(0)^2}E_6, && (87) \\
\partial_{S_1(0)}\partial_{S_2(0)}\tilde{h} &= \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)\tau S_1(0)S_2(0)}E_1 - \frac{1}{\sigma_1^2\tau S_1(0)S_2(0)}E_4 \\
&\quad + \frac{2\rho}{\sigma_1\sigma_2(1-\rho^2)\tau S_1(0)S_2(0)}E_6, && (88) \\
\partial_{S_2(0)}\partial_{S_2(0)}\tilde{h} &= \frac{-1}{\sigma_2^2(1-\rho^2)\tau S_2(0)^2}E_1 + \frac{1}{S_2(0)^2}E_5 \\
&\quad - \frac{2}{\sigma_2^2(1-\rho^2)\tau S_2(0)^2}E_6, && (89)
\end{aligned}
$$

where the terms E_i are defined by

$$
\begin{aligned}
E_1 &\triangleq \tilde{h}, && (90) \\
E_2 &\triangleq \tilde{h}\frac{-2A}{2\sigma_1^2\tau}, && (91) \\
E_3 &\triangleq \tilde{h}\frac{-A^2}{2\sigma_1^2\tau}, && (92) \\
E_4 &\triangleq \tilde{h}\frac{-2BA}{2\sigma_2^2(1-\rho^2)\tau}, && (93) \\
E_5 &\triangleq \tilde{h}\frac{-2B}{2\sigma_2^2(1-\rho^2)\tau}, && (94) \\
E_6 &\triangleq \tilde{h}\frac{-B^2}{2\sigma_2^2(1-\rho^2)\tau}. && (95)
\end{aligned}
$$

5.3 Conclusions

To obtain several relations between the Greeks one only has to do some linear algebra in $\mathbf{R}^6$. Some results are

$$
\begin{aligned}
0 &= \rho_{q^1} + S_1(0)\tau\Delta_1 && (96) \\
0 &= \rho_{q^2} + S_2(0)\tau\Delta_2 && (97) \\
0 &= q^1\rho_{q^1} + q^2\rho_{q^2} + \frac{1}{2}\sigma_1\Phi_1 + \frac{1}{2}\sigma_2\Phi_2 + r\rho_r + \tau\Theta && (98) \\
0 &= \Theta - rv + (r-q^1)S_1(0)\Delta_1 + (r-q^2)S_2(0)\Delta_2 \\
&\quad + \frac{1}{2}\sigma_1^2 S_1(0)^2 ,_{11} + \rho\sigma_1\sigma_2 S_1(0)S_2(0) ,_{12} + \frac{1}{2}\sigma_2^2 S_2(0)^2 ,_{22} && (99) \\
\kappa &= \sigma_1\sigma_2\tau S_1(0)S_2(0) ,_{12} && (100)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \rho\kappa - \sigma_1\Phi_1 + \sigma_1^2\tau S_1(0)^2 , _{11} && (101)\\
0 &= \rho\kappa - \sigma_2\Phi_2 + \sigma_2^2\tau S_2(0)^2 , _{22} && (102)\\
0 &= \sigma_1\Phi_1 - \sigma_2\Phi_2 - \sigma_1^2\tau S_1(0)^2 , _{11} + \sigma_2^2\tau S_2(0)^2 , _{22} && (103)\\
\rho_r &= -\tau\left(v - S_1(0)\Delta_1 - S_2(0)\Delta_2\right) && (104)\\
0 &= \tau v + \rho_{q^1} + \rho_{q^2} + \rho_r && (105)
\end{aligned}
$$

Of course one can get more relations by combining some relations above. The relations we have chosen to present are either similar to the one-dimensional case or have another natural interpretation.

- (96) and (97). These relations are a justification for the rough way to deal with dividends. One subtracts the dividends from the actual spot price and prices the option with this price and without dividends. This relation is not effected by the two-dimensionality of the problem.
- (98). This is the two-dimensional version of the general invariance under time scaling.
- (99). This is the Black-Scholes differential equation. This relation must hold, because we concentrated on European claims. It turns out, that the dynamic of an option price is described by the market model and that the price of the option is defined as a boundary problem.
- (100). The Greek κ, the change of the option price because of a change of the correlation was our motivation to do this calculation. One would expect a relationship between κ and $, _{12}$, but it is remarkable, that this relationship has such a simple structure.
- (101) and (102). Notice that one can determine the κ only by the knowledge of some derivatives with respect to parameters which concern only one stock. Of course, there is no difference between the first and the second stock. These relations are valid in the one-dimensional case with $\kappa \equiv 0$.
- (103). This is an extension of the vega-gamma relation as derived in the one-dimensional case, see (50).
- (104). The interest rate risk is well known to be the negative product of duration and the amount of money invested. The term in the parentheses is exactly the amount of money one would have to invest in the cash bond in order to delta-hedge the option.
- (105). This relation is the two-dimensional rates symmetry, an extension of equation (44).

5.4 Cross-Gamma and Correlation Risk

The simplicity of equation (100) is based on a well-known relationship of multivariate normal distribution functions published in [4], which says the following. We suppose that the vector X of n random variables with means zero and unit variances has a nonsingular normal multivariate distribution with probability density function

$$\phi_n(x_1,\ldots,x_n;c_{11},\ldots,c_{nn}) = (2\pi)^{-\frac{1}{2}n}|\mathbf{C}|^{\frac{1}{2}}\exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{C}\mathbf{x}\right). \tag{106}$$

Here $\mathbf{C}$ is the inverse of the covariance matrix of X, which is denoted by $\mathbf{R}$, and has elements $\{\rho_{ij}\}$. Then the following identity can be proved easily by writing the density in terms of its characteristic function.

$$\frac{\partial\phi_n}{\partial\rho_{ij}} = \frac{\partial^2\phi_n}{\partial x_i\partial x_j}. \tag{107}$$

In the two-dimensional case this reads as

$$\frac{\partial n_2(x,y;\rho)}{\partial\rho} = \frac{\partial^2 n_2(x,y;\rho)}{\partial x\partial y}, \tag{108}$$

which can be extended readily to the corresponding cumulative distribution function, i.e.,

$$\frac{\partial\mathcal{N}_2(x,y;\rho)}{\partial\rho} = \frac{\partial^2\mathcal{N}_2(x,y;\rho)}{\partial x\partial y} = n_2(x,y;\rho). \tag{109}$$

6 Examples in the Two-Dimensional Case

6.1 European Options on the Minimum/Maximum of Two Assets

We consider the payoff

$$\left[\phi\left(\eta\min(\eta S_T^{(1)},\eta S_T^{(2)}) - K\right)\right]^+. \tag{110}$$

This is a European put or call on the minimum ($\eta=+1$) or maximum ($\eta=-1$) of the two assets $S_T^{(1)}$ and $S_T^{(2)}$ with strike K. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put. Its value function has been published in [5] and can be written as

$$\begin{aligned}
& v(t,S_t^{(1)},S_t^{(2)},K,T,q^1,q^2,r,\sigma_1,\sigma_2,\rho,\phi,\eta) \\
= \quad & \phi\left[S_t^{(1)}e^{-q^1\tau}\mathcal{N}_2(\phi d_1,\eta d_3;\phi\eta\rho_1)\right. \\
+ \quad & S_t^{(2)}e^{-q^2\tau}\mathcal{N}_2(\phi d_2,\eta d_4;\phi\eta\rho_2) \\
- \quad & \left.Ke^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1-\sigma_1\sqrt{\tau}),\eta(d_2-\sigma_2\sqrt{\tau});\rho)\right)\right], \qquad (111)
\end{aligned}$$

$$\sigma^2 \triangleq \sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2, \tag{112}$$

$$\rho_1 \triangleq \frac{\rho\sigma_2 - \sigma_1}{\sigma}, \tag{113}$$

$$\rho_2 \triangleq \frac{\rho\sigma_1 - \sigma_2}{\sigma}, \tag{114}$$

$$d_1 \triangleq \frac{\ln(S_t^{(1)}/K) + (r - q^1 + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \tag{115}$$

$$d_2 \triangleq \frac{\ln(S_t^{(2)}/K) + (r - q^2 + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \tag{116}$$

$$d_3 \triangleq \frac{\ln(S_t^{(2)}/S_t^{(1)}) + (q^1 - q^2 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{117}$$

$$d_4 \triangleq \frac{\ln(S_t^{(1)}/S_t^{(2)}) + (q^2 - q^1 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{118}$$

6.1.1 Greeks

Space homogeneity implies that

$$v = S_t^{(1)}\frac{\partial v}{\partial S_t^{(1)}} + S_t^{(2)}\frac{\partial v}{\partial S_t^{(2)}} + K\frac{\partial v}{\partial K}. \tag{119}$$

Using this equation we can immediately write down the deltas

$$\frac{\partial v}{\partial S_t^{(1)}} = \phi e^{-q^1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1), \tag{120}$$

$$\frac{\partial v}{\partial S_t^{(2)}} = \phi e^{-q^2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2), \tag{121}$$

$$\frac{\partial v}{\partial K} = -\phi e^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho)\right). \tag{122}$$

Computing the gammas is actually the only situation where differentiation is needed. We use the identities

$$\frac{\partial}{\partial x}\mathcal{N}_2(x, y; \rho) = n(x)\mathcal{N}\left(\frac{y - \rho x}{\sqrt{1-\rho^2}}\right), \tag{123}$$

$$\frac{\partial}{\partial y}\mathcal{N}_2(x, y; \rho) = n(y)\mathcal{N}\left(\frac{x - \rho y}{\sqrt{1-\rho^2}}\right), \tag{124}$$

and obtain

$$\frac{\partial^2 v}{\partial (S_t^{(1)})^2} = \frac{\phi e^{-q^1\tau}}{S_t^{(1)}\sqrt{\tau}}\left[\frac{\phi}{\sigma_1}n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3 - d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right. \tag{125}$$

$$- \frac{\eta}{\sigma} n(d_3) \mathcal{N}\left(\phi\sigma \frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\Bigg],$$

$$\frac{\partial^2 v}{\partial (S_t^{(2)})^2} = \frac{\phi e^{-q^2\tau}}{S_t^{(2)}\sqrt{\tau}}\left[\frac{\phi}{\sigma_2} n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4 - d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right. \tag{126}$$

$$\left. - \frac{\eta}{\sigma} n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2 - d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right],$$

$$\frac{\partial^2 v}{\partial S_t^{(1)} \partial S_t^{(2)}} = \frac{\phi\eta e^{-q^1\tau}}{S_t^{(2)}\sigma\sqrt{\tau}} n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right). \tag{127}$$

The sensitivity with respect to correlation is directly related to the cross-gamma

$$\frac{\partial v}{\partial \rho} = \sigma_1\sigma_2\tau S_t^{(1)} S_t^{(2)} \frac{\partial^2 v}{\partial S_t^{(1)} \partial S_t^{(2)}}. \tag{128}$$

We refer to (101) and (102) to get the following formulas for the vegas,

$$\frac{\partial v}{\partial \sigma_1} = \frac{\rho v_\rho + \sigma_1^2 \tau (S_t^{(1)})^2 v_{S_t^{(1)} S_t^{(1)}}}{\sigma_1} \tag{129}$$

$$= S_t^{(1)} e^{-q^1\tau}\sqrt{\tau}\left[\rho_1\phi\eta n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right. \tag{130}$$

$$\left. + n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3 - d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right],$$

$$\frac{\partial v}{\partial \sigma_2} = \frac{\rho v_\rho + \sigma_2^2 \tau (S_t^{(2)})^2 v_{S_t^{(2)} S_t^{(2)}}}{\sigma_2} \tag{131}$$

$$= S_t^{(2)} e^{-q^2\tau}\sqrt{\tau}\left[\rho_2\phi\eta n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2 - d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right. \tag{132}$$

$$\left. + n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4 - d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right].$$

Looking at (96) and (97) the rhos are given by

$$\frac{\partial v}{\partial q^1} = -S_t^{(1)}\tau\frac{\partial v}{\partial S_t^{(1)}}, \tag{133}$$

$$\frac{\partial v}{\partial q^2} = -S_t^{(2)}\tau\frac{\partial v}{\partial S_t^{(2)}}, \tag{134}$$

$$\frac{\partial v}{\partial r} = -K\tau\frac{\partial v}{\partial K}. \tag{135}$$

Among the various ways to compute theta one may use the one based on (98).

$$\frac{\partial v}{\partial t} = -\frac{1}{\tau}\left[q^1 v_{q^1} + q^2 v_{q^2} + r v_r + \frac{\sigma_1}{2} v_{\sigma_1} + \frac{\sigma_2}{2} v_{\sigma_2}\right]. \tag{136}$$

6.2 Outside Barrier Options

The payoff of an outside barrier option is

$$[\phi\,(S_1(T) - K)]^+ \, I\!I_{\{\min_{0\le t\le T}(\eta S_2(t)) > \eta B\}} \tag{137}$$

This is a European put or call with strike K and a knock-out barrier H in a second asset, called the *outer* asset. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put and the binary variable η takes the value $+1$ for a lower barrier and -1 for an upper barrier. We abbreviate

$$\mu_i = r - q^i. \tag{138}$$

This is an example of a path-dependent rainbow option, a case which is not covered by the previous theory. However, it is illuminating to see how some of the ideas are still successfully applicable. The value provided by Heynen and Kat has been published in [3].

$$\begin{aligned}
V_0 &= \phi S_1(0) e^{-q^1 T} \mathcal{N}_2(\phi d_1, -\eta e_1; \phi\eta\rho) \\
&\quad -\phi S_1(0) e^{-q^1 T} \exp\left(\frac{2(\mu_2 + \rho\sigma_1\sigma_2)\ln(H/S_2(0))}{\sigma_2^2}\right) \mathcal{N}_2(\phi d_1', -\eta e_1'; \phi\eta\rho) \\
&\quad -\phi K e^{-rT} \mathcal{N}_2(\phi d_2, -\eta e_2; \phi\eta\rho) \\
&\quad +\phi K e^{-rT} \exp\left(\frac{2\mu_2 \ln(H/S_2(0))}{\sigma_2^2}\right) \mathcal{N}_2(\phi d_2', -\eta e_2'; \phi\eta\rho), && (139) \\
d_1 &= \frac{\ln(S_1(0)/K) + (\mu_1 + \sigma_1^2)T}{\sigma_1\sqrt{T}}, && (140) \\
d_2 &= d_1 - \sigma_1\sqrt{T}, && (141) \\
d_1' &= d_1 + \frac{2\rho\ln(H/S_2(0))}{\sigma_2\sqrt{T}}, && (142) \\
d_2' &= d_2 + \frac{2\rho\ln(H/S_2(0))}{\sigma_2\sqrt{T}}, && (143) \\
e_1 &= \frac{\ln(H/S_2(0)) - (\mu_2 + \rho\sigma_1\sigma_2)T}{\sigma_2\sqrt{T}}, && (144) \\
e_2 &= e_1 + \rho\sigma_1\sqrt{T}, && (145) \\
e_1' &= e_1 - \frac{2\ln(H/S_2(0))}{\sigma_2\sqrt{T}}, && (146) \\
e_2' &= e_2 - \frac{2\ln(H/S_2(0))}{\sigma_2\sqrt{T}}. && (147)
\end{aligned}$$

6.2.1 Greeks

Homogeneity in price tells us that

$$V_0 = S_1(0)\frac{\partial V_0}{\partial S_1(0)} + K\frac{\partial V_0}{\partial K}. \tag{148}$$

This allows us to read off the delta(inner spot)

$$\begin{aligned} \frac{\partial V_0}{\partial S_1(0)} &= \phi e^{-q^1 T}\mathcal{N}_2(\phi d_1, -\eta e_1; \phi\eta\rho) \\ &\quad -\phi e^{-q^1 T}\exp\left(\frac{2(\mu_2+\rho\sigma_1\sigma_2)\ln(H/S_2(0))}{\sigma_2^2}\right)\mathcal{N}_2(\phi d_1', -\eta e_1'; \phi\eta\rho) \end{aligned} \tag{149}$$

and the dual delta(inner strike)

$$\begin{aligned} \frac{\partial V_0}{\partial K} &= -\phi e^{-rT}\mathcal{N}_2(\phi d_2, -\eta e_2; \phi\eta\rho) \\ &\quad +\phi e^{-rT}\exp\left(\frac{2\mu_2\ln(H/S_2(0))}{\sigma_2^2}\right)\mathcal{N}_2(\phi d_2', -\eta e_2'; \phi\eta\rho). \end{aligned} \tag{150}$$

7 Generalization to Higher Dimensions and other Market Models

7.1 Multidimensional Black-Scholes Model

7.1.1 Scale Invariance of Time

The general idea presented in the one dimensional case is valid in higher dimensions too. Therefore we have got the relation, which hold for all $a > 0$:

$$\begin{aligned} &v(x_1, ..., x_n, \tau, r, q_1, ..., q_n, \sigma_{11}, ..., \sigma_{nn}) = \\ &v(x_1, ..., x_n, \frac{\tau}{a}, ar, aq_1, ..., aq_n, \sqrt{a}\sigma_{11}, ..., \sqrt{a}\sigma_{nn}) \end{aligned} \tag{151}$$

We differentiate with respect to a and evaluate at $a = 1$:

$$0 = \tau\Theta + r\rho + \sum_{i=1}^{n} q_i\rho_{q_i} + \frac{1}{2}\sum_{i,j=1}^{n}\Phi_{ij}\sigma_{ij} \tag{152}$$

Φ_{ij} denotes the differentiation of v with respect to σ_{ij}.

7.1.2 Price Homogeneity

Equations (39) and (40) easily extend to

$$v = \sum_{i=1}^{n} x_i\Delta_i + \sum_{j=1}^{m} k_j\Delta_j^k \tag{153}$$

$$\sum_{i,j=1}^{n} x_i x_j \, , \, {}_{ij} = \sum_{i,j=1}^{m} k_i k_j \, , \, {}^{k}_{ij} \tag{154}$$

for strike-defined options, and equations (41) and (42) to

$$0 = \sum_{i=1}^{n} x_i \Delta_i + \sum_{j=1}^{m} l_j \Delta^l_j \tag{155}$$

$$\sum_{i,j=1}^{n} x_i x_j \, , \, {}_{ij} + \sum_{i=1}^{n} x_i \Delta_i = \sum_{i,j=1}^{m} l_i l_j \, , \, {}^{l}_{ij} + \sum_{i=1}^{m} l_i \Delta^l_i \tag{156}$$

for level-defined options.

7.1.3 European Options in the Black-Scholes Model

The previous homogeneity based relations are model-independent. In the Black-Scholes model we may furthermore invoke the multidimensional Black-Scholes PDE

$$0 = -v_\tau - rv + \sum_{i=1}^{n} x_i (r - q_i) \partial_{x_i} v + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \circ \sigma^T)_{ij} x_i x_j \partial_{x_i} \partial_{x_j} v \tag{157}$$

to compute Greeks.

7.2 Beyond Black-Scholes

Up to now we illustrated our ideas in the Black-Scholes model and in some parts we used specific properties of this model. Nevertheless there are some properties, which are so fundamental, that they should hold in any realistic market model. These fundamental properties are the homogeneity of time, the scale invariance of time and the scale invariance of prices. For every market model one uses, one should check, if the model fulfills these properties.
An example for a market model with non-deterministic volatility is Heston's stochastic volatility model [2].
In this more general framework one needs to clarify the notion of vega, rho etc. A change of volatility could mean a change of the entire underlying volatility process. If the pricing formula depends on input parameters such as initial volatility, volatility of volatility, mean reversion of volatility, then one can consider derivatives with respect to such parameters. It turns out that our strategy to compute Greeks can still be applied successfully in a stochastic volatility model.

7.3 Heston's Stochastic Volatility Model

$$dS_t = S_t \left[\mu \, dt + \sqrt{v(t)} dW_t^{(1)} \right], \tag{158}$$

$$dv_t = \kappa(\theta - v_t) \, dt + \sigma \sqrt{v(t)} dW_t^{(2)}, \tag{159}$$

$$\begin{aligned}
\mathbf{Cov}\left[dW_t^{(1)}, dW_t^{(2)}\right] &= \rho\, dt, && (160)\\
\lambda(S,v,t) &= \lambda v. && (161)
\end{aligned}$$

The model for the variance v_t is the same as the one used by *Cox, Ingersoll and Ross* for the short term interest rate. We think of $\theta > 0$ as the long term variance, of $\kappa > 0$ as the rate of mean-reversion. The quantity $\lambda(S,v,t)$ is called the market price of volatility risk.
Heston provides a closed-form solution for European vanilla options paying

$$[\phi\,(S_T - K)]^+ . \qquad (162)$$

As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put, K the strike in units of the domestic currency, q the risk free rate of asset S, r the domestic risk free rate and T the expiration time in years.

7.3.1 Abbreviations

$$\begin{aligned}
a &\triangleq \kappa\theta && (163)\\
u_1 &\triangleq \frac{1}{2} && (164)\\
u_2 &\triangleq -\frac{1}{2} && (165)\\
b_1 &\triangleq \kappa + \lambda - \sigma\rho && (166)\\
b_2 &\triangleq \kappa + \lambda && (167)\\
d_j &\triangleq \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)} && (168)\\
g_j &\triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{b_j - \rho\sigma\varphi i - d_j} && (169)\\
\tau &\triangleq T - t && (170)\\
D_j(\tau,\varphi) &\triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{\sigma^2}\left[\frac{1-e^{d_j\tau}}{1-g_j e^{d_j\tau}}\right] && (171)\\
C_j(\tau,\varphi) &\triangleq (r-q)\varphi i\tau && (172)\\
&\quad + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\varphi i + d)\tau - 2\ln\left[\frac{1-g_j e^{d_j\tau}}{1-e^{d_j\tau}}\right]\right\}\\
x = \ln S_t & && (173)\\
f_j(x,v,t,\varphi) &\triangleq e^{C_j(\tau,\varphi)+D_j(\tau,\varphi)v+i\varphi x} && (174)\\
P_j(x,v,\tau,y) &\triangleq \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\varphi y} f_j(x,v,\tau,\varphi)}{i\varphi}\right] d\varphi && (175)\\
p_j(x,v,\tau,y) &\triangleq \frac{1}{\pi}\int_0^\infty \Re\left[e^{-i\varphi y} f_j(x,v,\tau,\varphi)\right] d\varphi && (176)
\end{aligned}$$

$$P_+(\phi) \quad \triangleq \quad \frac{1-\phi}{2} + \phi P_1(\ln S_t, v_t, \tau, \ln K) \tag{177}$$

$$P_-(\phi) \quad \triangleq \quad \frac{1-\phi}{2} + \phi P_2(\ln S_t, v_t, \tau, \ln K) \tag{178}$$

This notation is motivated by the fact that the numbers P_j are the cumulative distribution functions (in the variable y) of the log-spot price after time τ starting at x for some drift μ. The numbers p_j are the respective densities.

7.3.2 Value

The value function for European vanilla options is given by

$$V = \phi \left[e^{-q\tau} S_t P_+(\phi) - K e^{-r\tau} P_-(\phi) \right] \tag{179}$$

The value function takes the form of the Black-Scholes formula for vanilla options. The probabilities $P_\pm(\phi)$ correspond to $\mathcal{N}(\phi d_\pm)$ in the constant volatility case.

7.3.3 Greeks

The deltas can be obtained based on the homogeneity of prices.

Spot delta.

$$\Delta \triangleq \frac{\partial V}{\partial S_t} = \phi e^{-q\tau} P_+(\phi) \tag{180}$$

Dual delta.

$$\frac{\partial V}{\partial K} = -\phi e^{-r\tau} P_-(\phi) \tag{181}$$

Gamma.

$$, \triangleq \frac{\partial \Delta}{\partial S_t} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial S_t} = \frac{e^{-q\tau}}{S_t} p_1(\ln S_t, v_t, \tau, \ln K) \tag{182}$$

As in the case of vanilla options we observe that

$$S_t e^{-q\tau} p_1(\ln S_t, v_t, \tau, \ln K) = K e^{-r\tau} p_2(\ln S_t, v_t, \tau, \ln K), \tag{183}$$

Rho. Rho is connected to delta via equations (48) and (49).

$$\frac{\partial V}{\partial r} = \phi K e^{-r\tau} \tau P_-(\phi), \tag{184}$$

$$\frac{\partial V}{\partial q} = -\phi S_t e^{-q\tau} \tau P_+(\phi). \tag{185}$$

Theta. Theta can be computed using the partial differential equation for the Heston vanilla option

$$V_t + (r-q)SV_S + \frac{1}{2}\sigma v V_{vv} + \frac{1}{2}vS^2V_{SS} + \rho\sigma v S V_{vS} - qV$$
$$+[\kappa(\theta - v) - \lambda]V_v = 0, \tag{186}$$

where the derivatives with respect to initial variance v must be evaluated numerically.

8 Summary

We have learned how to employ homogeneity-based methods to compute analytical formulas of Greeks for analytically known value functions of options in a one-and higher-dimensional market. Restricting the view to the Black-Scholes model there are numerous relations between various Greeks which are of fundamental interest. The method helps saving computation time for the mathematician who has to differentiate complicated formulas as well as for the computer, because analytical results for Greeks are usually faster to evaluate than finite differences involving at least twice the computation of the option's value. Knowing how the Greeks are related to each other can speed up finite-difference-, tree-, or Monte Carlo-based computation of Greeks or lead at least to a quality check. Many of the results are valid beyond the Black-Scholes model. Most remarkably some relations of the Greeks are based on properties of the normal distribution refreshing the active interplay between mathematics and financial markets to our very pleasure.

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