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Abstract

We investigate the characteristic functions of multi-factor Cheyette Models and the application to the valuation of interest rate derivatives. The model dynamic can be classified as an affine-diffusion process implying an exponential structure of the characteristic function. The characteristic function is determined by a model specific system of ODEs, that can be solved explicitly for arbitrary Cheyette Models. The necessary transform inversion turns out to be numerically stable as a singularity can be removed. Thus the pricing methodology is reliable and we use it for the calibration of multi-factor Cheyette Models to caps.

Keywords: Cheyette Model, Characteristic Function, Fourier Transform, Calibration of Multi-Factor Models

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1 Introduction

In 1992, D. Heath, R. Jarrow and A. Morton (HJM) (Heath, Jarrow & Morton 1992) have developed a general framework to model the dynamics of the entire forward rate curve in an interest rate market. The associated valuation approach is based on mainly two assumptions: the first one postulates, that it is not possible to gain riskless profit (No-arbitrage condition), and the second one assumes the completeness of the financial market. The HJM model, or strictly speaking the HJM framework, is a general model environment and incorporates many previously developed models like the Vasicek model (1977) (Vasicek 1977) or the Hull-White model (1990) (Hull & White 1990). The general setting mainly suffers from two disadvantages: first of all the difficulty to apply the model in market practice and second, the extensive computational complexity caused by the high-dimensional stochastic process of the underlying. The first disadvantage was improved by the development of the LIBOR Market Model (1997) introduced by (Brace, Gatarek & Musiela 1997), (Jamshidian 1997) and (Miltersen & Sandmann 1997), which combines the general risk-neutral yield curve model with market standards. The second disadvantage can be improved by restricting the general HJM model to a subset of models with a similar specification of the volatility structure. The resulting system of Stochastic Differential Equations (SDE) describing the yield curve dynamic breaks down from a high-dimensional process into a low-dimensional structure of Markovian processes. Furthermore, the dependence on the current state of the process allows the valuation by a certain Partial Differential Equation (PDE). This approach was developed by O. Cheyette in 1994 (Cheyette 1994).

The Cheyette Models are factorial models, that means multi-factor models can be constructed easily as canonical extensions of one-factor models. In practice, the Cheyette Models usually incorporate several factors to achieve sufficient flexibility to represent the market state. The model dynamic considers all factors and might become a high-dimensional SDE as each factor captures one dimension. The price of interest rate derivatives is given as the expected value of the terminal payoff under a given model dynamic. Thus, the computation comes up to a multi-dimensional integral. If one knows the probability density function of the random variable representing the model dynamic, the multi-dimensional integral can be transformed to a one-dimensional one. In particular, the dimension is independent of the number of factors incorporated in the model. Unfortunately, the probability density function seldom exists in closed-form, but its Fourier Transform is often known explicitly.

The Fourier Transform of the probability density function is known as the *characteristic function*. Based on the Inverse Fourier Transform of the characteristic function one can compute the expected value of a given function, e.g. the final payoff function of a derivative, under a certain model. If one knows the characteristic function, the (numerical) pricing of derivatives becomes less complex, because the computation of the expected value of the payoff function reduces to a one dimensional complex integral. In their work, Duffie, Pan and Singleton (Duffie, Pan & Singleton 1999) showed, that the characteristic function of a general affine jump diffusion process (AJD) X_t has an exponential structure

$$\exp[A(t, T, u) + B(t, T, u)X_t].$$

The characteristic function is fully specified by determining the functions $A(t, T, u)$ and $B(t, T, u)$ given as unique solutions to a system of complex valued ordinary differential equations (ODEs). The affine jump diffusion process X_t is defined as the solution to the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathrm{d}Z_t,$$

where W_t denotes an standard Brownian motion and Z_t a pure jump process. Further it is assumed, that the drift μ and the volatility σ hold an affine structure. The Cheyette Model can be classified in this framework. The special structure of the Cheyette Models simplifies the system of ODEs and allows to compute the functions A and B explicitly. Consequently, the pricing setup can be applied to Cheyette Models and in particular we can value interest rate options. The valuation of interest rate derivatives is fast, e.g. the valuation of a single cap takes about 10^{-3} sec. CPU time[1]. This valuation method can for example be used to calibrate multi-factor Cheyette Models to the market state.

The numerical tractability is analyzed in this paper and we show, that

[1]We used a Windows based PC with Intel Core 2 Duo CPU @ 1.66 GHz and 3.25 GB RAM.

the computation of the integral is stable as we can remove a singularity of first order.

At the beginning of the paper, we give a short introduction of the structure of Cheyette Model and embed it in the general AJD framework. The theoretical background is followed by the construction of characteristic functions and some applications to the Ho-Lee and the exponential Hull-White Model. In the following we will verify the theoretical results by some numerical application of cap pricing. Finally, we investigate the numerical tractability, in particular of the transform inversion, which turns out to be straightforward.

2 Literature Review

The application of Fourier Transforms for pricing derivatives is a well established method that is still en vogue for current research. The application of this technique in finance was initialized by Heston (Heston 1993), who searched a relationship between the characteristic function of the pricing kernel of the underlying asset and the pricing formula. In the last years, mainly two further approaches by Carr and Madan (1999) and Lewis (2001) have been established. Carr and Madan (Carr & Madan 1999) introduced a technique to represent the price of an option in terms of a Fourier Transform. Therefore, they performed the Fourier Transform of the payoff function with respect to the strike. Thus the transform can be substituted in the pricing integral and after changing the integration order, one achieves the price as a function of the characteristic function of the density. In contrast, Lewis (Lewis 2001) set up the Fourier Transform with respect to the underlying asset. Thereby, Lewis could separate the Fourier Transform of the payoff from the transform of the pricing kernel. Thus, he introduced a more general setup, that is valid for a broad spectrum of payoff functions.

The technique presented in this paper can be assigned to the approach of Duffie and Kan (Duffie & Kan 1996). They first established the link between affine stochastic processes and exponential affine term structure models. In particular, they showed, that the factor coefficients of these term structure models are solutions to a system of simultaneous Riccati equations. This approach was further explored and applied to interest rate option pricing by Duffie, Pan and Singleton (Duffie et al. 1999). Similar constructions can be found in the works of Bakshi and Madan (Bakshi & Madan 2000) and Cherubini (Cherubini 2009).

3 Risk-Neutral Pricing and the Forward Measure

The intended application of the characteristic function is the pricing of interest rate derivatives. Therefore we apply the risk-neutral pricing framework, which guarantees arbitrage-free markets. In the following we are working on a probability space $(\Omega, \mathfrak{F}, P)$ and according to the setup as exemplarily presented in (Shreve 2004), the price of a derivative security $V(t)$ at time $t > 0$ is given by

$$V(t) = \widetilde{\mathbb{E}}[\exp\left(-\int_t^T R(u)du\right) V(T)|\mathfrak{F}_t] \quad , 0 \leq t \leq T,$$

where $\widetilde{\mathbb{E}}[\ .\ |\mathfrak{F}_t]$ denotes the conditional expectation with respect to the risk-neutral measure $\widetilde{P}$.

Definition 3.1 (Risk-Neutral Measure).
A probability measure $\widetilde{P}$ is said to be risk-neutral *if*

(i) $\widetilde{P}$ and P are equivalent and

(ii) under $\widetilde{P}$, the discounted asset prices are martingales.

The basic motivation why we use risk-neutral measures is given by the fundamental theorems of asset pricing as presented in (Shreve 2004).

Theorem 3.2 (First fundamental theorem of asset pricing).
If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Theorem 3.3 (Second fundamental theorem of asset pricing).
Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

The definition of a risk-neutral measure is linked to the choice of numéraire, which is the unit of assets in which other assets are denominated. The dynamic of a model is given with respect to a specified measure and thus it depends on the choice of numéraire. Changing the perspective slightly, one can use a change of numéraire to change the modeling considerations. Depending on the choice of numéraire, the model can be complicated or simple. In principle, any positively priced asset can be taken as numéraire, but we shall take any non-dividend-paying asset.

In the following we will use the zero-coupon-bond price $B(t,T)$ as numéraire. This reference is only valid or existing up to time $T \geq t$. Therefore it can be applied only to value claims which are paid up to time T. The associated martingale measure is called the *time T-forward measure* abbreviated by Q^T. This measure is called T-forward measure, because the forward price of some payoff X at time T is the expectation of X under the time T-forward measure. In other words, the T-forward prices are martingales under the T-forward measure Q^T.

4 The Cheyette Model

Assume $B(t,T)$ to be the time t price of a zero-coupon bond maturing at time $T \geq t$. The usual continuously compounded forward rate at time t for deposit is given by

$$f(t,T) = \frac{-\partial \ln B(t,T)}{\partial T}.$$

Heath, Jarrow and Morton (Heath et al. 1992) showed, that in any arbitrage-free term structure model with continuous evaluation of the yield curve the forward rate has to satisfy

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T) \left(\int_s^T \sigma(s,v)\mathrm{d}v \right) ds + \int_0^t \sigma(s,T)\mathrm{d}W(s),$$

where W is a Brownian motion under the risk-neutral measure. The model is fully specified by a given volatility structure $\{\sigma(t,T)\}_{T\geq t}$ and the initial forward curve. The class of Cheyette interest rate models, first presented in (Cheyette 1994), forms a subset of the general class of HJM models. As already suggested in the literature, one can choose a specific volatility structure $\sigma(t,T)$ and achieves an exogenous model of the yield curve with Markovian dynamics. We will follow the ansatz of O. Cheyette (Cheyette 1994) and use a separable volatility term structure. The volatility function is assumed to be separable into time and maturity dependent factors. The volatility function is parameterized by a finite sum of separable functions

$$\sigma(t,T) = \sum_{i=1}^{N} \alpha_i(T) \frac{\beta_i(t)}{\alpha_i(t)}. \tag{1}$$

The choice of the volatility structure affects the characteristic of the models, (Beyna & Wystup 2010). The dynamic of the forward rate can be reformulated as follows, if we assume the mentioned volatility structure:

$$f(t,T) = f(0,T) + \sum_{j=1}^{N} \frac{\alpha_j(T)}{\alpha_j(t)} \left[x_j(t) + \sum_{i=1}^{N} \frac{A_i(T) - A_i(t)}{\alpha_i(t)} V_{ij}(t) \right]. \tag{2}$$

The expression uses the following notation for $i, j = 1, ..., N$:

$$\begin{aligned}
A_k(t) &= \int_0^t \alpha_k(s) \mathrm{d}s, \\
x_i(t) &= \int_0^t \frac{\alpha_i(t)}{\alpha_i(s)} \beta_i(s)\, \mathrm{d}W(s) + \\
&\quad \int_0^t \frac{\alpha_i(t)\beta_i(s)}{\alpha_i(s)} \left[\sum_{k=1}^{N} \frac{A_k(t) - A_k(s)}{\alpha_k(s)} \beta_k(s) \right] \mathrm{d}s, \\
V_{ij}(t) &= V_{ji}(t) = \int_0^t \frac{\alpha_i(t)\alpha_j(t)}{\alpha_i(s)\alpha_j(s)} \beta_i(s)\beta_j(s) \mathrm{d}s.
\end{aligned}$$

The dynamic of the forward rate in a one-factor model is determined by the state variables $x_i(t)$ and $V_{ij}(t)$ for $i, j = 1, ..., N$. The stochastic variable x_i describes the short rate and the non-stochastic variable V_{ij} states the cumulative quadratic variation. Summarizing, the forward rate is determined by $\frac{N}{2}(N+3)$ state variables. The dynamics of the short rate and the quadratic variation are given by Markov processes as

$$\begin{aligned}
\mathrm{d}x_i(t) &= \left(x_i(t)\partial_t(\log \alpha_i(t)) + \sum_{k=1}^{N} V_{ik}(t) \right) \mathrm{d}t + \beta_i(t)\mathrm{d}W(t) \\
\frac{\mathrm{d}}{\mathrm{d}t} V_{ij}(t) &= \beta_i(t)\beta_j(t) + V_{ij}(t)\partial_t(\log(\alpha_i(t)\alpha_j(t))).
\end{aligned}$$

The Cheyette Models are factorial models and thus, they can be generalized easily to multi-factor models. The additional factors are given by several independent Brownian motions and the forward rate is given by

$$f(t,T) = f(0,T) + \sum_{i=1}^{M} \tilde{f}^i(t,T),$$

where $\tilde{f}^i(t,T)$ denotes a one factor forward rate defined by (2).

5 Affine Diffusion Setup

5.1 Fundamentals

The valuation of financial securities in an arbitrage-free environment incorporates the trade-off between analytical and numerical tractability of pricing and the complexity of the probability model for the state variable X. Thus many academics and practioners impose structure on the conditional distribution of X to obtain closed- or nearly closed-form expressions. Following the idea of Duffie, Pan and Singleton (Duffie et al. 1999), we assume that X follows an affine diffusion process (AD). This assumption appears to be particularly efficient in developing tractable, dynamic asset pricing models. The affine diffusion process is a specialization of the affine jump-diffusion process (AJD), that build the basis for the Gaussian Vasicek model (Vasicek 1977) or the Cox, Ingersoll and Ross model (Cox, Ingersoll & Ross 1985). The application to the class of Cheyette models does not require jumps in the dynamic and therefore the limitation is reasonable.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space with filtration $\mathfrak{F}_t$. We assume that X is a Markov process relative to $\mathfrak{F}_t$ in some state space $D \subset \mathbb{R}^n$ solving the stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{3}$$

where W denotes a $\mathfrak{F}_t$ -standard Brownian Motion in $\mathbb{R}^n$. In the following we impose an affine structure on the drift $\mu : D \to \mathbb{R}$, the volatility $\sigma : D \to \mathbb{R}^{n\times n}$ and the associated discount rate $R : D \to \mathbb{R}$:

1. $\mu(x) = K_0 + K_1 x$, for $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$,
2. $[\sigma(x)\sigma(x)^T]_{ij} = (H_0)_{ij} + (H_1)_{ij} x$, for $H = (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$,

3. $R(x) = \rho_0 + \rho_1 x$, for $\rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n$.

Definition 5.1.
We define the characteristic χ *of a random variable* X *as the tuple of coefficients incorporated in the affine structure* $\chi = (K, H, \rho)$.

The characteristic χ determines the distribution of a random variable X completely, if the initial condition $X_0 = X(0)$ is given, and it captures the effects of any discounting.

5.2 Classification of the Cheyette Model

The class of Cheyette models is part of the general affine diffusion framework. In order to express the Cheyette model in terms of the affine diffusion notation, we have to specify the characteristic of the state variable X. According to the model design presented in Section 4 and by using the introduced notations, the drift $\mu : D \to \mathbb{R}^n$ is given by

$$[\mu(x)]_i = \partial_t(\log \alpha_i(t))x_i(t) + \sum_{k=1}^{N} V_{ik}(t),$$

where the index i denotes the i-th component. Thus, the coefficient K is specified as

$$(K_0)_i = \sum_{k=1}^{N} V_{ik}(t),$$

$$\begin{aligned}(K_1)_{ij} &= \begin{cases} \partial_t \log \alpha_i(t), & i \neq j \\ 0, & i = j \end{cases} \\ &= \begin{cases} \frac{\partial_t \alpha_i(t)}{\alpha_i(t)}, & i \neq j \\ 0, & i = j. \end{cases}\end{aligned}$$

The matrix K_1 is a diagonal matrix with entries $\frac{\partial_t \alpha_i(t)}{\alpha_i(t)}$ on the diagonal and zeros otherwise. The coefficients representing the volatility

$$[\sigma(x)\sigma(x)^T]_{ij} = \beta_i(t)\beta_j(t)$$

turn out to be

$$\begin{aligned}[H_0]_{ij} =& \beta_i(t)\beta_j(t),\\ (H_1) =& 0.\end{aligned}$$

The coefficients of the affine structure of the discount rate

$$R(x) = f(0,t) + \sum_{k=1}^{N} x_k(t)$$

are determined in a similar manner as

$$\begin{aligned}\rho_0 &= f(0,t),\\ \rho_1 &= 1,\end{aligned}$$

where $f(0,t)$ denotes the initial forward rate up to time $t > 0$. Therefore, the characteristic of the state variable X of the general Cheyette model is specified. Furthermore, we assume an initial condition $X(0) = 0$ and thus, the distribution of the random variable is fully determined.

6 Characteristic Functions

6.1 Fundamentals

The stochastic dynamics of the forward rate are described by the distributions of some random variables, known as state variables. According to basic probability theory the distributions are represented by their density functions, which are rarely available in closed form. Alternatively, the density function can be fully characterized by its Fourier Transform, which is known as its *characteristic function*. The Fourier Transform $F(y)$ of a function $f(x)$ is defined as

$$F(y) = \int_{-\infty}^{\infty} f(x)\exp(\imath xy)\mathrm{d}x, \tag{4}$$

where $\imath$ denotes the imaginary unit, (Lukacs 1970). Theoretically, the Fourier Transform is a generalization of the complex Fourier Series in the limit as the function period tends to infinity. There exist several common conventions in the definition of the Fourier Transform. According to the definition of the Fourier Transform, its inverse is defined as

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(y) \exp(-\imath xy) \mathrm{d}y.$$

The density function can be achieved by applying the inverse transform to the characteristic function.

6.2 Characteristic Functions in the Affine Diffusion Setup

In the following we will use a slightly different transform to define the characteristic function in the context of affine diffusion processes, which was first suggested by Duffie, Pan and Singleton (Duffie et al. 1999). The transform is an extension of the introduced Fourier Transform (4) with discounting at rate $R(X_t)$. Based on the characteristic χ the transform

$$\psi^{\chi} : \mathbb{C}^n \times D \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$$

of X_T conditional on $\mathfrak{F}_t$ when well defined at $t \leq T$ is given by

$$\psi^{\chi}(u, X_t, t, T) = E^{\chi} \left[\exp\left(- \int\limits_t^T R(X_s) ds \right) \exp(uX_T) \, | \mathfrak{F}_t \right], \tag{5}$$

where E^{χ} denotes expectation under the distribution of X determined by χ. The definition of the transform ψ^{χ} differs from the normal (conditional) characteristic function of the distribution of X_T by the discounting at rate $R(X_t)$.

In their work, Duffie et al. (Duffie et al. 1999) showed, that under some technical regularity conditions, the transform has an exponential shape and is determined completely by solutions to a system of ordinary differential equations. The transform depends on the characteristic χ and is given by

$$\psi^{\chi}(u, x, t, T) = \exp\left[A(t) + B(t)x\right], \tag{6}$$

where $A(t)$ and $B(t)$ satisfy complex valued ordinary differential equations (ODEs)

$$\dot{B}(t) = \rho_1 - K_1^T B(t) - \frac{1}{2} B(t)^T H_1 B(t), \tag{7}$$

$$\dot{A}(t) = \rho_0 - K_0 B(t) - \frac{1}{2} B(t)^T H_0 B(t), \tag{8}$$

with boundary conditions

$$B(T) = u, \tag{9}$$

$$A(T) = 0. \tag{10}$$

Remark 6.1.
The system of ODEs results straightforward from an application of Ito's Formula to $\psi^{\chi}(u, x, t, T) = \exp(A(t) + B(t)x)$.

The regularity conditions on the characteristic, that makes the transform well defined are given by the following definition.

Definition 6.2.
A characteristic $\chi = (K, H, \rho)$ *is* well-behaved *at* $(u, T) \in \mathbb{C}^n \times [0, \infty)$ *if the corresponding system of ODEs* (7) *-* (10) *is solved uniquely by* A *and* B *and if the following conditions are fulfilled:*

(i) $E\left[\left(\int_0^T \eta_t \eta_t dt\right)^{\frac{1}{2}}\right] < \infty$,

(ii) $E[|\Psi_T|] < \infty$,

where

$$\Psi_t = \exp\left(-\int_0^t R(X_s) ds\right) \exp(A(t) + B(t)x(t))$$

and

$$\eta_t = \Psi_t B(t)^T \sigma(X_t).$$

Theorem 6.3.
Suppose the characteristic $\chi = (K, H, \rho)$ *is well-behaved at* (u, T)*. Then the transform* ψ^{χ} *of* X *defined by (5) is given by (6).*

The proof of this general theorem is given in (Duffie et al. 1999). The dynamic of the model and the associated characteristic depends on the choice of numéraire. As already presented in Section 3, we set up the pricing of interest rate derivatives with respect to the T-forward measure. Thus we need to perform a change of measure as the original model consideration typically assumes the money market account as numéraire $N_t = \exp(rt)$ with risk-free interest rate r. The associated equivalent martingale measure Q^N is risk-neutral and must be translated to the T-forward measure Q^T. Consequently, the model dynamics change and so does the characteristic. The Radon-Nikodyn derivative characterizes the change of measure and can be calculated explicitly. The effect of the change of measure on the characteristic and the implied Fourier-Transform can be quoted in dependence of the Radon-Nikodyn derivative as presented in (Duffie et al. 1999). The description of the change of measure or the equivalent change of numéraire is most suitable by the following theorem.

Theorem 6.4 (Change of numéraire)**.**
Assume Q^N and Q^M to be risk-neutral probability measures with respect to the numéraires N_t and M_t. The Radon-Nikodyn derivative that changes the measure Q^M into Q^N is given by

$$\frac{\mathrm{d}Q^N}{\mathrm{d}Q^M} = \frac{\frac{N_T}{N_t}}{\frac{M_T}{M_t}}.$$

In the following we assume the Radon-Nikodyn derivative

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{\xi_T}{\xi_0}$$

to define an equivalent probability measure where

$$\xi_t = \exp\left(-\int_0^t R(X_s)ds\right)\exp\left[\tilde{\alpha}(t,T,b) + \tilde{\beta}(t,T,b)X_t\right]. \tag{11}$$

The characteristic under this change of measure is defined in the following proposition:

Proposition 6.5 (Transform under change of measure).
Assume $\chi^P = (K^P, H^P, \rho^P)$ *to be the characteristic associated to the probability measure* P. *The characteristic* $\chi^Q = (K^Q, H^Q, \rho^Q)$ *is associated to the probability measure* Q *and is created by the use of the Radon-Nikodyn derivative*

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{\xi_T}{\xi_0}.$$

The characteristic χ^Q *is defined by*

- $K_0^Q(t) = K_0^P(t) + H_0^P(t)\tilde{\beta}(t, T, b)$,
- $K_1^Q(t) = K_1^P(t) + H_1^P(t)\tilde{\beta}(t, T, b)$,
- $H^Q(t) = H^P(t)$,
- $\rho^Q = \rho^P$.

According to the intended pricing setup, we need to change the measure from the risk-neutral measure Q^N with numéraire $N_t = \exp(\int_t^T r(s)\mathrm{d}s)$ to the T-forward measure Q^T with the zero-coupon-bond price as numéraire $M_t = \frac{1}{B(t,T)}$. In the style of the change of measure Theorem 6.4, the Radon-Nikodyn derivative is defined by

$$\begin{aligned}
\frac{dQ^T}{dQ^N} &= \frac{\frac{M_T}{M_t}}{\frac{N_T}{N_t}} \\
&= \frac{\frac{1}{P(t,T)}}{\exp\left(\int_t^T r(s)\mathrm{d}s\right)}.
\end{aligned}$$

The price of the zero-coupon-bond at time t can be expressed in terms of the characteristic function by

$$\begin{aligned}
P(t, T) &= \mathbb{E}^\chi\Big[\exp\left(-\int_t^T r(s)\mathrm{d}s\right)\Big|X_t\Big] \\
&= \Psi^\chi(0, X_t, t, T) \\
&= \exp\left(A(t, T, 0) + B(t, T, 0)X_t\right).
\end{aligned}$$

In other words, the price of the zero-coupon-bond is given by the characteristic function Ψ^{χ} that is created with the boundary condition $u = 0$ in the fundamental ordinary differential equations. The implied T-forward measure is defined by the Radon-Nikodyn derivative

$$\frac{dQ^T}{dQ^N} = \exp\left[-A(t,T,0) - B(t,T,0)X_t)\right] \exp\left(-\int_t^T r(s)ds\right).$$

Consequently, the density function ξ_t, that determines the transform under the change of measure in Proposition 6.5 is defined for the change from the risk-neutral measure to the T-forward measure by

$$\tilde{\alpha}(t,T,b) = -A(t,T,0) \tag{12}$$

$$\tilde{\beta}(t,T,b) = -B(t,T,0). \tag{13}$$

Summarizing, we showed how to perform a change of measure in the framework of characteristic functions. Furthermore we stated the effect on the characteristic and defined the elements explicitly. Finally we demonstrated the method exemplarily for the change from the risk-neutral measure with the money market account as numéraire to the T-forward measure associated to the zero-coupon-bond price as numéraire.

6.3 Characteristic Functions in the Cheyette Model

In the previous section, we introduced the general framework for characteristic functions in the affine diffusion setup. The class of Cheyette Models can be integrated in this general setup as done in Section 6.2. In the following, we will clarify the construction of the characteristic function by calculating them in concrete models. We focus on the Ho-Lee Model and the exponential Hull-White Model exemplarily for one-factor models. Furthermore, we will focus on multi-factor models and present the implementation in an exponential model.

6.3.1 One Factor Models

6.3.1.1 Ho-Lee Model The Ho-Lee Model introduced by (Ho & Lee 1986) is the simplest one-factor model in the class of Cheyette models. The volatility is assumed to be constant

$$\sigma(t,T) = c$$

thus the dimension of the state space equals $n = 1$. In terms of the Cheyette Model introduced in Section 4, the volatility $\sigma(t,T) = \beta(t)\frac{\alpha(T)}{\alpha(t)}$ is determined by

$$\begin{aligned}\alpha(t) &= 1,\\ \beta(t) &= c.\end{aligned}$$

The dynamic of the state variable is based on the function $V(t)$ as presented in Section 5.2. In the Ho-Lee model it is given by

$$\begin{aligned}V(t) &= \int_0^t \frac{\alpha(t)^2}{\alpha(s)}\beta(s)^2 \mathrm{d}s\\ &= tc^2.\end{aligned}$$

Thus the characteristic $\chi^Q = (K^Q, H^Q, \rho^Q)$ with respect to the risk-neutral measure Q representing the dynamic of the model as introduced in Section 5.2 is given by

$$\begin{aligned}K_0^Q(t) &= V(t) = tc^2\\ K_1^Q(t) &= \frac{\partial_t \alpha(t)}{\alpha(t)} = 0\\ H_0^Q(t) &= \beta(t) = c^2\\ H_1^Q(t) &= 0\\ \rho_0(t) &= f\\ \rho_1(t) &= 1,\end{aligned}$$

where $f = f(0,T)$ denotes the initial forward rate (assumed to be constant). The characteristic function is given in dependance of the functions $A(t,T,u)$ and $B(t,T,u)$ defined as (unique) solutions to a system of ordinary differential equations, see Section 6.2. In the Ho-Lee Model the ODEs are given by

$$\begin{aligned}\dot{B}(t) &= 1,\\ \dot{A}(t) &= f - tc^2B(t) - \frac{1}{2}c^2B(t)^2,\end{aligned}$$

with boundary values

$$\begin{aligned}B(T) &= u,\\ A(T) &= 0.\end{aligned}$$

This system of ODEs is (uniquely) solved by

$$\begin{aligned}B(t) &= u + t - T,\\ A(t) &= -\frac{1}{2}c^2t^3 + c^2(T-u)t^2 + t(c^2uT - f - \frac{1}{2}u^2c^2)\\ &\quad + fT + \frac{1}{2}c^2u^2T - \frac{u^2}{2}T^3.\end{aligned}$$

In order to price interest rate derivatives, we have to change the measure to the T-forward measure as presented in Section 6.2. The Radon-Nikodyn derivative is determined by (12) and (13). The change of measure influences the dynamic of the model and consequently the associated characteristic. The characteristic χ^{Q^T} is associated to the T-forward measure Q^T and can be calculated by

$$\begin{aligned}K_0^{Q^T}(t) &= K_0^Q(t) + H_0^Q\tilde{\beta}(t,T,u)\\ &= V(t) - c^2B(t,T,0)\\ &= tc^2 - c^2(t-T),\end{aligned}$$

where $B(t,T,0)$ denotes the solution to the ODEs with zero-boundary values associated to the characteristic χ^Q. Similarly,

$$\begin{aligned}K_1^{Q^T}(t) &= K_1^Q(t) + H_1^Q(t)\tilde{\beta}(t,T,u)\\ &= 0.\end{aligned}$$

The remaining components of the characteristic χ^Q stay invariant under the change of measure,

$$H^{Q^T}(t) = H^Q(t),$$
$$\rho^{Q^T}(t) = \rho^Q(t).$$

In order to calculate the characteristic function we have to build up the system of ODEs based on χ^{Q^T} and solve it,

$$\dot{B}(t) = 1,$$
$$\dot{A}(t) = f - [tc^2 - c^2(t-T)](u+t-T) - \frac{c^2}{2}(u+t-T)^2$$

with boundary conditions

$$B(T) = u,$$
$$A(T) = 0.$$

The system is solved uniquely by

$$B(t) = u + t - T,$$
$$A(t) = f(t-T) - \frac{c^2}{6}(t-T)[t^2 + tT - 2T^2 + 3u(t+T) + 3u^2].$$

These functions determine the characteristic function in the Ho-Lee Model with respect to the T-forward measure.

6.3.1.2 Exponential Hull-White Model The exponential Hull-White Model is specified by the volatility

$$\sigma(t,T) = c\exp[-(T-t)\kappa].$$

In terms of the Cheyette Model introduced in Section 4, the volatility $\sigma(t,T) = \beta(t)\frac{\alpha(T)}{\alpha(t)}$ is determined by

$$\alpha(t) = \exp(-t\kappa),$$
$$\beta(t) = c.$$

The dynamic of the state variable is based on the function $V(t)$ as presented in Section 5.2. In the exponential Hull-White model it is given by

$$\begin{aligned} V(t) &= \int_0^t \frac{\alpha(t)^2}{\alpha(s)^2}\beta(s)^2 \mathrm{d}s \\ &= \int_0^t \exp\left[-2\kappa(t-s)\right] c^2 \mathrm{d}s \\ &= \frac{-c^2(-1+\exp(-2\kappa t))}{2\kappa}. \end{aligned}$$

Furthermore, we need the following quantity to determine the characteristic

$$\begin{aligned} \frac{\partial_t \alpha(t)}{\alpha(t)} &= \frac{-\kappa \exp(-t\kappa)}{\exp(-t\kappa)} \\ &= -\kappa. \end{aligned}$$

Thus the characteristic $\chi^Q = (K^Q, H^Q, \rho^Q)$ with respect to the risk-neutral measure Q representing the dynamic of the model as introduced in Section 5.2 is given by

$$\begin{aligned} K_0^Q(t) &= V(t) = \frac{-c^2(-1+\exp(-2\kappa t))}{2\kappa}, \\ K_1^Q(t) &= \frac{\partial_t \alpha(t)}{\alpha(t)} = -\kappa, \\ H_0^Q(t) &= \beta(t) = c^2, \\ H_1^Q(t) &= 0, \\ \rho_0(t) &= f, \\ \rho_1(t) &= 1. \end{aligned}$$

The characteristic function is given in dependance of the functions $A(t,T,u)$ and $B(t,T,u)$ defined as (unique) solutions to a system of ordinary differential equations, see Section 6.2. In the exponential Hull-White Model the ODEs are given by

$$\begin{aligned}
\dot{B}(t) &= 1 + \kappa B(t), \\
\dot{A}(t) &= f - V(t)B(t) - \frac{1}{2}c^2 B(t)^2,
\end{aligned}$$

with boundary values

$$\begin{aligned}
B(T) &= u, \\
A(T) &= 0.
\end{aligned}$$

This system of ODEs is (uniquely) solved by

$$\begin{aligned}
B(t) =& \exp(\kappa(t-T))\left[u - \frac{-1+\exp(c(T-t))}{\kappa}\right], \\
A(t) =& f(t-T) + \frac{c^2}{2\kappa^2}\left[1 + \frac{1}{c-\kappa} - \frac{\exp(-2\kappa T)}{c+\kappa}\right. \\
& + \exp[c(T-t) - \kappa(t+T)]\left[\frac{\exp(2\kappa t)}{\kappa - c} + \frac{1}{c+\kappa}\right] + u \\
& - \exp[\kappa(t-T)](1+u) + \exp(-2\kappa T)(1+u) \\
& \left. - \exp[-\kappa(t+T)](1+u)\right] \\
& + \frac{c}{4\kappa^2}\left[3 + \exp(2c(T-t)) + 4\kappa u\right. \\
& \left. - 4\exp(c(T-t))(1+\kappa u) - 2c(t-T)(1+\kappa u)^2\right]
\end{aligned}$$

In order to price interest rate derivatives, we have to change the measure to the T-forward measure as presented in Section 6.2. The Radon-Nikodyn derivative is determined by (12) and (13). The change of measure influences the dynamic of the model and consequently the associated characteristic. The characteristic χ^{Q^T} is associated to the T-forward measure Q^T and can be calculated by

$$
\begin{aligned}
K_0^{Q^T}(t) =& K_0^Q(t) + H_0^Q \tilde{\beta}(t,T,u) \\
=& V(t) - c^2 B(t,T,0) \\
=& -\frac{c^2(-1+\exp(-2\kappa t))}{2\kappa} \\
& - c^2 \exp(\kappa(t-T)) \left[-\frac{-1+\exp(c(T-t))}{\kappa} \right],
\end{aligned}
$$

where $B(t,T,0)$ denotes the solution to the ODEs associated to the characteristic χ^Q. Furthermore,

$$
\begin{aligned}
K_1^{Q^T}(t) &= K_1^Q(t) + H_1^Q(t)\tilde{\beta}(t,T,u) \\
&= -\kappa.
\end{aligned}
$$

The remaining components of the characteristic χ^Q stay invariant under the change of measure,

$$
\begin{aligned}
H^{Q^T}(t) &= H^Q(t), \\
\rho^{Q^T}(t) &= \rho^Q(t).
\end{aligned}
$$

In order to calculate the characteristic function we have to build up the system of ODEs based on χ^{Q^T} and solve it.

$$
\begin{aligned}
\dot{B}(t) =& 1 + \kappa B(t), \\
\dot{A}(t) =& f + \left[\frac{c^2(-1+\exp[-2\kappa t])}{2\kappa} + c^2 \exp[\kappa(t-T)](-\frac{-1+\exp[c(T-t)]}{\kappa}) \right] \\
& \exp(\kappa(t-T)) \left[u - \frac{-1+\exp[c(T-t)]}{\kappa} \right] \\
& - \frac{c^2}{2} \exp[2\kappa(t-T)] \left[u - \frac{-1+\exp[c(T-t)]}{\kappa} \right]^2 \\
=& f + \left[u - \frac{-1+\exp[c(T-t)]}{\kappa} \right] \exp(\kappa(t-T)) \\
& \left[\frac{c^2(-1+\exp[-2\kappa(t-T)])}{2\kappa} - uc^2 \exp(2\kappa(t-T)) \right] \\
& + \frac{c^2}{2} \exp[2\kappa(t-T)] \left[u - \frac{-1+\exp[c(T-t)]}{\kappa} \right]^2
\end{aligned}
$$

with boundary conditions

$$\begin{aligned}B(T) &= u,\\ A(T) &= 0.\end{aligned}$$

The system is solved uniquely by

$$\begin{aligned}
B(t) =& \exp(\kappa(t-T)) \left[u - \frac{-1+\exp(c(T-t))}{\kappa}\right],\\
A(t) =& f(t-T) + \frac{c^2}{12\kappa^3}\Bigg[3 + \frac{3\kappa\exp[-2(c-\kappa)(t-T)] + 6\kappa\exp[-(c-\kappa)(T-t)]}{\kappa - c}\\
&+ \frac{6\exp[c(T-t)-\kappa(t+T)]}{c+\kappa} - \frac{12\kappa^2 u\exp[-(c-3\kappa)(t-T)]}{c-3\kappa}\\
&- 6(1+\kappa u)\Big[\exp(\kappa(t-T)) + \exp(-\kappa(t+T))\Big]\\
&+ \frac{12\kappa(1+\kappa u)\exp(-(c-2\kappa)(t-T))}{c-2\kappa} - 4\exp[3\kappa(t-T)]\kappa u(1+\kappa u)\\
&+ 3\exp[2\kappa(t-T)](1+\kappa u)^2 + \frac{6\exp(-2\kappa T)(c+\kappa u(c+\kappa))}{c+\kappa}\\
&+ \kappa\left[\frac{-12}{c-2\kappa} + \frac{9}{c-\kappa} + u\left(4+\kappa\left(\frac{12\kappa}{c^2-5c\kappa+6\kappa^2}+u\right)\right)\right]\Bigg].
\end{aligned}$$

These functions determine the characteristic function in the exponential Hull-White Model with respect to the T-forward measure.

6.3.2 Multi Factor Models

The Cheyette interest rate models are factor models implying that multi-factor models can be constructed canonically out of one-factor models. As presented in Section 4 the forward rate $f(t,T)$ in the multi-factor model is given by

$$f(t,T) = f(0,T) + \sum_{k=1}^{M} f^k(t,T),$$

where $f^k(t,T)$ denotes the forward rate of the k-th one factor model and $f(0,T)$ denotes the initial value. Each one factor model is completely determined by the volatility function parameterized according to (1) by

$$\sigma^k(t,T) = \sum_{i=1}^{N_k} \frac{\alpha_i^k(T)}{\alpha_i^k(t)} \beta_i^k(t).$$

Thus it contains N_k state variables. If the multi-factor model incorporates M factors, the model is described by $n = \sum_{k=1}^{M} N_k$ state variables. In other words, the state space is n dimensional.

The characteristic function is defined in dependance of the characteristic χ introduced in Section 5.2 for arbitrary dimensions. The implied system of ODEs (7) - (8) stays unchanged. The general Cheyette Model prescribes the shape of the coefficients. Especially the structure of $H_1 = 0$ and the diagonal structure of matrix K_1 simplifies the calculation of the solutions.

$$\begin{aligned}
\dot{B}(t) =& \rho_1 - K_1(t)^T B(t) - \frac{1}{2} B(t)^T H_1(t) B(t) \\
=& \rho_1 - K_1(t)^T B(t) \\
=& [\rho_1]_i - \sum_{j=1}^{n} [K_1(t)]_{ij} B_j(t) \quad \text{(per component)} \\
=& [\rho_1]_i - [K_1(t)]_{ii} B_i(t).
\end{aligned}$$

First, the term of second order in the ODE disappears in consequence of $H_0 = 0$. Second, the n dimensional system of first order is decoupled thanks to the diagonal structure of the matrix $K_1(t)$. As a consequence the i-th component of $B(t) \in \mathbb{R}^n$ is no longer linked to the j-th $(i \neq j)$ component. Thus, the solution $B(t)$ can be calculated separately in every dimension. We would like to emphasize, that this simplification is just based on the structure of the coefficients in the general Cheyette Model and does not require further assumptions. In practice, the calculation of the characteristic functions in the multi-factor model can be traced back to the one dimensional case $(n = 1)$. In the following, we will demonstrate the calculation of the characteristic function exemplarily in a three factor model proposed by Cheyette (Cheyette

1994). The volatility functions are parameterized by

$$
\begin{aligned}
\sigma^{(1)}(t,T) =& c_1 + \beta_1^1 \exp(-\kappa_1^1(T-t)) + \beta_2^1 \exp(-\kappa_2^1(T-t)),\\
\sigma^{(2)}(t,T) =& c_2 + \beta_1^2 \exp(-\kappa_1^2(T-t)) + \beta_2^2 \exp(-\kappa_2^2(T-t)),\\
\sigma^{(3)}(t,T) =& c_3 + \beta_1^3 \exp(-\kappa_1^3(T-t)) + \beta_2^3 \exp(-\kappa_2^3(T-t))\\
&+ \beta_3^3 \exp(-\kappa_3^3(T-t)).
\end{aligned}
$$

The first and second factor require $N_i = 5$ state variables each, c_i, β_1^i, β_2^i, κ_1^i and κ_2^i for $i = 1,2$. The third factor incorporates 7 state variables, c_3, β_1^3, β_2^3, β_3^3, κ_1^3, κ_2^3 and κ_3^3. Thus, the dimension of the state space equals $n = \sum_{k=1}^{M} N_k = 17$. Each dimension relates to one summand of the volatility function $\sigma^{(i)}(t,T)$. The choice of parametrization implies that each component can either be traced back to the Ho-Lee Model or the exponential Hull-White Model. We investigated the construction of the characteristic function of these one-factor models separately in Section 6.3.1. Concerning the coefficients of the characteristic we have to distinguish between the constants c_i and the exponential terms $\beta_i^j \exp[-\kappa_i^j(T-t)]$. In the following we will assume that the first three components represent the constant terms c_i and the last 14 components correspond to the exponential function. Thus, the characteristic with respect to the risk-neutral measure Q is given by the following parameters $K_0^Q(t) \in \mathbb{R}^{19}$, $K_1^Q(t) \in \mathbb{R}^{19\times 19}$, $H_0^Q(t) \in \mathbb{R}^{19\times 19}$, $\rho_0 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}^{19}$:

$$
K_0^Q(t) = \begin{pmatrix}
t(c_1)^2\\
t(c_2)^2\\
t(c_3)^2\\
-(\beta_1^1)^2\Big(-1+\exp[-2\kappa_1^1 t]\Big)\\
\vdots\\
-(\beta_3^3)^2\Big(-1+\exp[-2\kappa_3^3 t]\Big)
\end{pmatrix}
$$

$$K_1^Q(t) = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & 0 & \\ & & & -\kappa_1^1 & & \\ & & 0 & & \ddots & \\ & & & & & -\kappa_3^3 \end{pmatrix}$$

$$\begin{aligned} H_0(t) &= \beta_i(t)\beta_j(t) \\ &= \begin{pmatrix} c_1^2 & c_2c_1 & c_3c_1 & \beta_1^1 c_1 & \cdots & \beta_3^3 c_1 \\ c_1c_2 & c_2^2 & c_3c_2 & \beta_1^1 c_2 & \cdots & \beta_3^3 c_2 \\ c_1c_3 & c_2c_3 & c_3^2 & \beta_1^1 c_3 & \cdots & \beta_3^3 c_3 \\ c_1\beta_1^1 & \cdots & & (\beta_1^1)^2 & & \vdots \\ \vdots & & & & \ddots & \\ c_1\beta_3^3 & & \cdots & & & (\beta_3^3)^2 \end{pmatrix} \\ H_1 &= 0 \\ \rho_0 &= f \\ \rho_1 &= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

The function $B(t)$ solving the ODE (7) can be solved separately per component. Applying the results of Section 6.3.1, $B(t)$ is given by

$$B_i(t) = \begin{cases} u_i + t - T, & i = 1,2,3 \\ \exp(\kappa_i(t-T))\left[u_i - \frac{-1+\exp(\beta_i(T-t))}{\kappa_i}\right], & i = 4,\dots,17. \end{cases}$$

The ODEs defining $A(t)$ is given by

$$\begin{aligned} \dot{A}(t) =& \rho_0 - K_0(t)B(t) - \frac{1}{2}B(t)^T H_0(t)B(t) \\ =& f - \sum_{i=1}^{n}[K_0(t)]_i B_i(t) - \frac{1}{2}\sum_{i=1}^{n} B_i(t)\left(\sum_{j=1}^{n}[H_0(t)]_{ij}B_j(t)\right). \end{aligned}$$

The unique solution is given by

$$A(t) = \int_T^t f - \sum_{i=1}^n [K_0(x)]_i B_i(x) - \frac{1}{2} \sum_{i=1}^n B_i(x) \left(\sum_{j=1}^n [H_0(x)]_{ij} B_j(x) \right) \mathrm{d}x.$$

The function can be computed explicitly, but in this case it becomes unmanageable and we evaluate it numerically.

In the last section, we showed how to construct the characteristic function for multi-factor models. This function can be computed explicitly, but unfortunately it becomes extensive. Thus we have to evaluate it by simple numerical integration methods.

Finally, we will show, that the characteristics in the general Cheyette model are well-behaved, which is a necessary condition for the pricing with characteristic functions.

Theorem 6.6.
The characteristics $\chi = (K, H, \rho)$ of the Cheyette Model are well-behaved at $(u, T) \in \mathbb{C}^n \times [0, \infty)$.

Proof.
The well-behavior can be proved by verifying the conditions of Definition 6.2. First, we have to show, that the system of ODEs (7) - (8) can be solved uniquely. As presented in Section 6.3.2, the system of ODEs defining the function $B(t)$ can be decoupled and solved separately in every dimension. Thus, each ODE is an inhomogeneous ordinary differential equation of first order with initial values. According to (Walter 2000), each ODE can be solved uniquely, if the coefficient functions are continuous. These functions are determined by the characteristic, that consists of affine functions as presented in Section 5.2. Thus, these linear functions are continuous and consequently the ODEs can be solved uniquely. The ODE determining function $A(t)$ is treated in the same way.

In addition to the unique existence, we have to verify the conditions

(i) $E\left[\left(\int_0^T \eta_t \eta_t dt \right)^{\frac{1}{2}} \right] < \infty,$

(ii) $E\Big[\ |\Psi_T| \ \Big] < \infty,$

where

$$\Psi_t = \exp\left(-\int_0^t R(X_s)ds\right)\exp(A(t)+B(t)x(t))$$

and

$$\eta_t = \Psi_t B(t)^T \sigma(X_t),$$

to prove the well-behavior of the characteristic. The finiteness of both expressions is implied directly, if we could show, that a unique solution to the stochastic differential equation (3) exists in the Lebesgue space $L^2(D)$ with state space $D \subset \mathbb{R}^n$. Therefore, we will apply the *Existence and Uniqueness Theorem* published in (Evans 2003) and repeated in the Appendix A.2. Thus, we have to verify that the drift $\mu(x,t)$ and the volatility $\sigma(x,t)$ are uniformly Lipschitz continuous in the variable x. First we focus on the drift

$$\mu : D \to \mathbb{R}, \text{ for } D \subset \mathbb{R}^n.$$

$$\begin{aligned}
|\mu(x,t) - \mu(\hat{x},t)| =& |K_0 + K_1 x - K_0 - K_1\hat{x}| \\
=& |K_1(x-\hat{x})| \\
\leq& |K_1||x-\hat{x}|
\end{aligned}$$

$$\begin{aligned}
|\mu(x,t)| =& |K_0 + K_1 x| \\
\leq& |K_0| + |K_1||x| \\
\leq& L(1+|x|),
\end{aligned}$$

where $L := \max(|K_0|, |K_1|)$.
Second we focus on the volatility $\sigma(x,t) = \beta(t)$.

$$\begin{aligned}
|\sigma(x,t) - \sigma(\hat{x},t)| &= |\beta(t) - \beta(t)| \\
&= 0
\end{aligned}$$

$$\begin{aligned}
|\sigma(x,t)| &= |\beta(t)| \\
&\leq |\beta(t)|(1+|x|)
\end{aligned}$$

So far, we verified the first two conditions of the uniqueness and existence theorem. The initial value X_0 is given by $X_0 = 0$ in the Cheyette Model. Thus the remaining assumptions are fulfilled and therefore we showed the unique existence of a solution to the SDE and that this solution is in $L^2(D)$, which completes the proof. □

7 Pricing with Characteristic Functions

7.1 Fundamentals

The fundamental idea of this paper is the usage of characteristic functions to price interest rate derivatives, especially options. The setup as presented in (Duffie et al. 1999) can in particular be used to price derivatives with a payoff

$$(\exp(a + dX_T) - c)^+$$

paid at time T with initial condition X_0. The price of this general claim with respect to the characteristic χ at time $t = 0$ is given by

$$\begin{aligned}
\Gamma(X_0, a, d, c, T) &= \mathbb{E}^{\chi}\left[\exp\left(-\int_0^T R(X_s)\mathrm{d}s\right)[\exp(a + dX_T) - c]^+\right] \\
&= \mathbb{E}^{\chi}\left[\exp\left(-\int_0^T R(X_s)\mathrm{d}s\right)[\exp(a + dX_T) - c]\mathbb{1}_{\{\exp(a+dX_T)>c\}}\right] \\
&= \mathbb{E}^{\chi}\left[\exp\left(-\int_0^T R(X_s)\mathrm{d}s\right)[\exp(a + dX_T) - c]\mathbb{1}_{\{-dX_T \le a - \ln(c)\}}\right].
\end{aligned}$$

This representation can be expressed in terms of the inverse Fourier-Transform of the characteristic function. Therefore, we introduce

$$G_{a,b}(., x, T, \chi) : \mathbb{R} \to \mathbb{R}_+$$

defined by

$$G_{a,b}(y, X_0, T, \chi) = \mathbb{E}^{\chi}\left[\exp\left(-\int_0^T R(X_s)\mathrm{d}s\right)\exp(aX_T)\mathbb{1}_{\{bX_T \leq y\}}\right]. \quad (14)$$

Thus, the price of the general claim can be expressed as follows:

$$\begin{aligned}\Gamma(X_0, a, d, c, T) \\ &= \mathbb{E}^{\chi}\left[\exp\left(-\int_0^T R(X_s)\mathrm{d}s\right)(\exp(a + dX_T) - c)\mathbb{1}_{\{-dX_T \leq a - \ln(c)\}}\right] \\ &= \exp(a)\Big[G_{d,-d}(a - \ln(c), X_0, T, \chi) \\ &\quad - \exp(-a)cG_{0,-d}(a - \ln(c), X_0, T, \chi)\Big].\end{aligned}$$

The Fourier Transform $\widehat{G}_{a,b}(., X_0, T, \chi)$ of $G_{a,b}(., X_0, T, \chi)$ is given by

$$\begin{aligned}\widehat{G}_{a,b}(v, X_0, T, \chi) &= \int_{\mathbb{R}} \exp(\imath vy)\mathrm{d}G_{a,b}(y, X_0, T, \chi) \\ &= \mathbb{E}^{\chi}\left[\exp\left(-\int_0^T R(X_s)\mathrm{d}s\right)\exp[(a + \imath vb)X_T]\right] \\ &= \Psi^{\chi}(a + \imath vb, X_0, 0, T).\end{aligned}$$

The values of $G_{a,b}(., X_0, T, \chi)$ can be obtained by inverting the Fourier-Transform of the characteristic function. This calculation shows explicitly the influence of the characteristic function on the pricing of interest rate derivatives.

Proposition 7.1 (Transform inversion).
Suppose for fixed $T \in [0, \infty)$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, that the characteristic $\chi = (K, H, \rho)$ is well-behaved at $(a + \imath vb, T)$ for any $v \in \mathbb{R}$ and suppose

$$\int_{\mathbb{R}} |\Psi^{\chi}(a + \imath vb, x, 0, T)| \mathrm{d}v < \infty.$$

Then $G_{a,b}(., x, T, \chi)$ is well-defined and given by

$$\begin{aligned} G_{a,b}(y, X_0, T, \chi) = & \frac{\Psi^{\chi}(a, X_0, 0, T)}{2} \\ & - \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{Im}\left[\Psi^{\chi}(a + \imath vb, X_0, 0, T) \exp(-\imath vy)\right]}{v} \mathrm{d}v. \end{aligned}$$

The proof of the proposition is given in Appendix A.2. It is mainly based on the ideas of Duffie et al. (Duffie et al. 1999), but contains some adjustments.

The integrand of the transform inversion has a singularity in $v = 0$ of order 1. We will show in Section 8, that this singularity is removable. The limit $v \to 0$ exists and one can compute the values explicitly.

As already presented in Section 6.2 we have to build up the characteristic function based on the T-forward measure Q^T. The associated characteristic will be named χ^{Q^T} in the following. The change of the characteristic implies some changes in the pricing formulas as well. The change of measure is mainly driven by the density function

$$\xi_t = \exp[\alpha(t, T, b) + \beta(t, T, b) X_t]$$

determining the Radon-Nikodyn derivative. Duffie et al. showed in (Duffie et al. 1999) that the pricing formula needs to be adjusted. Following their ideas, the price of a general claim with respect to the new measure Q at time $t = 0$ is given by

$$\begin{aligned} \Gamma(X_0, a, d, c, T) = & \exp[a + \tilde{\alpha}(T, T, b)] \Big[G_{\tilde{\beta}(T,T,b)+d,-d}\Big(a - \ln(c), X_0, T, \chi^Q\Big) \\ & - \exp(-a) c G_{\tilde{\beta}(T,T,b),-d}\Big(a - \ln(c), X_0, T, \chi^Q\Big)\Big]. \end{aligned} \tag{15}$$

In case of the change from the risk-neutral measure Q^N to the T-forward measure Q^T implemented by (12) and (13) the price is given by

$$\begin{aligned}\Gamma(X_0, a, d, c, T)\\ =&\exp(a - A(T,T,0))\Big[G_{-B(T,T,0)+d,-d}\Big(a - \ln(c), X_0, T, \chi^{Q^T}\Big)\\ &- \exp(-a)cG_{-B(T,T,0),-d}\Big(a - \ln(c), X_0, T, \chi^{Q^T}\Big)\Big],\end{aligned} \tag{16}$$

where $A(t,T,0)$ and $B(t,T,0)$ denote the solutions to the ordinary differential equations with zero boundary values associated to the characteristic χ^{Q^T}.

7.2 Cap / Floor

An interest rate cap is a derivative that provides insurance against the floating rate of interest rises above a certain level. Let $\tau, 2\tau, ..., n\tau$ be the fixed dates for future interest payments. At each fixed date $\kappa\tau$, the interest rate is capped at $\bar{r} \in \mathbb{R}$ and the cap leads to a payoff at time $\kappa\tau$ of

$$L\tau\Big[\mathcal{R}((\kappa-1)\tau, \kappa\tau) - \bar{r}\Big]^+$$

where L denotes the nominal amount and $\mathcal{R}[(\kappa-1)\tau, \kappa\tau]$ the τ-year floating interest rate at time $(\kappa-1)\tau$ defined by

$$\frac{1}{1 + \tau\mathcal{R}[(\kappa-1)\tau, \kappa\tau]} = \Lambda\Big((\kappa-1)\tau, \kappa\tau\Big).$$

The time-T market price of a zero-coupon bond maturing at time $s > T$ is given by $\Lambda(T,s)$. It can easily expressed in terms of the characteristic function $\Psi^{\chi}(u, X_t, t, T)$

$$\begin{aligned}\Lambda(T,s) =&\exp[A(T,s,0) + B(T,s,0)X_T]\\ =&\Psi^{\chi}(0, X_T, T, s),\end{aligned}$$

where $A(t,T,u)$ and $B(t,T,u)$ denotes the solutions to (7) and (8). The market value at time $t = 0$ of the cap paying at date $\kappa\tau$ can be expressed as

$$
\begin{aligned}
Cap(\kappa) =& \mathbb{E}^Q\Big[\exp\Big(-\int_0^{\kappa\tau} R(X_u)\mathrm{d}u\Big)\,\tau\Big(\mathcal{R}[(\kappa-1)\tau,\kappa\tau]-\bar{r}\Big)^+\Big] \\
=& (1+\tau\bar{r})\mathbb{E}^Q\Big[\exp\Big(-\int_0^{(\kappa-1)\tau} R(X_u)\mathrm{d}u\Big) \\
& \Big(\frac{1}{1+\tau\bar{r}}-\Lambda((\kappa-1)\tau,\kappa\tau)\Big)^+\Big].
\end{aligned}
$$

Thus the pricing of the $cap(\kappa)$ is equivalent to the pricing of a put option starting in $(\kappa-1)\tau$ and matures in $\kappa\tau$ with strike $\frac{1}{1+\tau\bar{r}}$. This transformation is based on the ideas published by Hull (Hull 2005). Exploiting the put-call parity, the price of the cap at time $t=0$ is in accordance to (Duffie et al. 1999) given by

$$
\begin{aligned}
Cap(\kappa) =& (1+\tau\bar{r})\Big[\Gamma(X_0,\bar{A},\bar{B},\frac{1}{1+\tau\bar{r}},(\kappa-1)\tau) \\
& -\Lambda(0,\kappa\tau)+\frac{\Lambda(0,(\kappa-1)\tau)}{1+\tau\bar{r}}\Big], \qquad (17)
\end{aligned}
$$

where $\Gamma(X_0,a,d,c,T)$ is the price of a claim with payoff $(\exp(a+dX_T)-c)^+$ at time T, $\bar{A}=A((\kappa-1)\tau,\kappa\tau,0)$ and $\bar{B}=B((\kappa-1)\tau,\kappa\tau,0)$. The pricing is set up with respect to the risk-neutral measure Q and the associated characteristic χ^Q. The functions $A(t,T,u)$ and $B(t,T,u)$ result from the system of ordinary differential equations (7) and (8).

An interest rate floor is a derivative that provides a payoff when the underlying floating interest rate falls below a certain level. In analogy to caps, a floor can be seen as a call option on the interest rate and one receives similar pricing formulas as in the case of caps. The pricing formula for caps and floors can be evaluated by the help of characteristic functions. In Section 7 we showed how to compute the market values of a general claim. Applying this valuation formula to (17), one reaches the pricing formula for caps by using characteristic functions:

$$
\begin{aligned}
Cap(\kappa) = & (1+\tau\overline{r})\Bigg[\frac{\exp[A(0,(\kappa-1)\tau,0)+B(0,(\kappa-1)\tau,0)X_0]}{1+\tau\overline{r}} \\
& - \exp[A(0,\kappa\tau,0)+B(0,\kappa\tau,0)X_0] \\
& + \exp[A((\kappa-1)\tau,\kappa\tau,0)) - A((\kappa-1)\tau,(\kappa-1)\tau,0)\Bigg] \\
& \Big\{G_{-B\left((\kappa-1)\tau,(\kappa-1)\tau,0\right)+B\left((\kappa-1)\tau,\kappa\tau,0\right),-B\left((\kappa-1)\tau,\kappa\tau,0\right)} \\
& \Big(A((\kappa-1)\tau,\kappa\tau,0)-\ln(\frac{1}{1+\tau\overline{r}}),X_0,T,\chi^{Q^T}\Big) \\
& - \exp(-A((\kappa-1)\tau,\kappa\tau,0))\frac{1}{1+\tau\overline{r}} \\
& G_{-B\left((\kappa-1)\tau,(\kappa-1)\tau,0\right),-B\left((\kappa-1)\tau,\kappa\tau,0\right)} \\
& \Big(A((\kappa-1)\tau,\kappa\tau,0)-\ln(\frac{1}{1+\tau\overline{r}}),X_0,T,\chi^{Q^T}\Big)\Big\}
\end{aligned} \tag{18}
$$

We applied this pricing formula and in the following section we present the results including a verification of the method.

7.3 Quality Check

Until now we developed the theoretical framework for characteristic functions. This includes on the one hand the construction and on the other hand the application to pricing derivatives. Thereby we incorporated arbitrary numbers of factors in the model. In addition to the theoretical development we will give a practical justification of the construction and an evidence for the correct implementation. Therefore, we will price several caps by characteristic functions and compare the results to the prices obtained by semi-closed formulas developed by Henrard (Henrard 2003). The pricing formulas by Henrard are limited to one factor models only. The derivation and the application to Cheyette Models is substantially analyzed in (Beyna & Wystup 2010). In order to produce comparable results we restrict the quality check to one factor models. The analysis is subdivided into the following steps:

1. Compute cap prices in the Black-Scholes model from existing market data, e.g. the implied volatility σ_{impl}.
2. Calibrate the Ho-Lee model to cap prices and obtain a (unique) vola-

tility σ^{Ch} in the Cheyette Model.

3. Compute cap prices by characteristic functions in the Ho-Lee model using the volatility σ^{Ch}.

4. Compute the implied volatility σ^{Ch}_{impl} in the Black-Scholes model from the cap price obtained by characteristic functions.

5. Compare the original implied volatility σ_{impl} to the implied volatility σ^{Ch}_{impl} obtained from pricing with characteristic functions.

The relevant measure for the quality of the pricing is the difference in the implied (Black-Scholes) volatility. This measure delivers a standardized criterion as it is independent of the moneyness, the level of volatility and the remaining lifetime. The computation of the prices based on characteristic functions includes some numerical integration for the transform inversion in Theorem 7.1. Therefore we tested several methods and the choice influences the accuracy, stability and speed of the price computation. The numerical behavior of the computation is analyzed in Section 8. The results presented in the following are based on the generalized Gauss-Laguerre quadrature with weights $w(x) = \exp(-x)x^2$ and 150 supporting points, (Press 2002).

The quality examination covers 20 caps with varying lifetime, moneyness and implied volatility. Coming from an initial interest rate of 7% the strikes change between 6% (in-the-money), 7% (at-the-money) and 8% (out-of-the-money). In the first part we assume a implied volatility of 20% and shift the starting time of the cap. We focus on caps starting in 1, 2, 3, 4 and 5 years and mature one year later. The results are summarized in Table 1 and are illustrated in Figure 1, Figure 2 and Figure 3. In addition we plotted the differences in implied volatility in Figure 4.

The results show that the prices computed by the characteristic function match the prices in the Black-Scholes model, which equal the prices computed by semi-closed formulas according to (Henrard 2003). The error in differences in implied volatility is small and varies between the minimum of -0.1178% and the maximum of 0.1136%. The average of the signed differences is 0.0015% and the average of the absolute differences amount to 0.0604%. Furthermore there is no noticeable trend in the error, like a systematic over or under valuation. In 7 of 15 cases (47%) the characteristic

Implied Volatility	Strike	Start Time	Maturity	BS-Price	CF-Price	Price differences	Differences in implied volatility
0.2	0.08	1.0	2.0	0.2357	0.2361	0.0004	0.0137%
0.2	0.08	2.0	3.0	0.3998	0.4031	0.0023	0.0702%
0.2	0.08	3.0	4.0	0.5071	0.5036	-0.0035	-0.0963%
0.2	0.08	4.0	5.0	0.5798	0.5818	0.0020	0.0500%
0.2	0.08	5.0	6.0	0.6290	0.6338	0.0048	0.1136%
0.2	0.07	1.0	2.0	0.5586	0.5558	-0.0028	-0.1178%
0.2	0.07	2.0	3.0	0.7083	0.7081	-0.0002	-0.0060%
0.2	0.07	3.0	4.0	0.7940	0.7974	0.0034	0.0936%
0.2	0.07	4.0	5.0	0.8449	0.8435	-0.0014	-0.0364%
0.2	0.07	5.0	6.0	0.8733	0.8773	0.0040	0.0996%
0.2	0.06	1.0	2.0	1.1198	1.1198	0.0000	-0.0001%
0.2	0.06	2.0	3.0	1.1769	1.1754	-0.0015	-0.0614%
0.2	0.06	3.0	4.0	1.2033	1.2002	-0.0031	-0.1060%
0.2	0.06	4.0	5.0	1.2087	1.2082	-0.0005	-0.0179%
0.2	0.06	5.0	6.0	1.1998	1.2006	0.0008	0.0234%
0.10	0.08	3.0	4.0	0.1542	0.1532	-0.0010	0.0317%
0.15	0.08	3.0	4.0	0.3250	0.3252	0.0002	0.0047%
0.20	0.08	3.0	4.0	0.5071	0.5036	-0.0035	-0.0963%
0.25	0.08	3.0	4.0	0.6929	0.6947	0.0017	0.0461%
0.30	0.08	3.0	4.0	0.8796	0.8834	0.0038	0.1015%

Table 1: Presentation of the results of pricing caps by characteristic functions (in the Ho-Lee Model) and in the Black-Scholes Model. Further, we state the errors in terms of differences in implied volatility. The parameters like the strike, the starting time and the (original) implied volatility change.

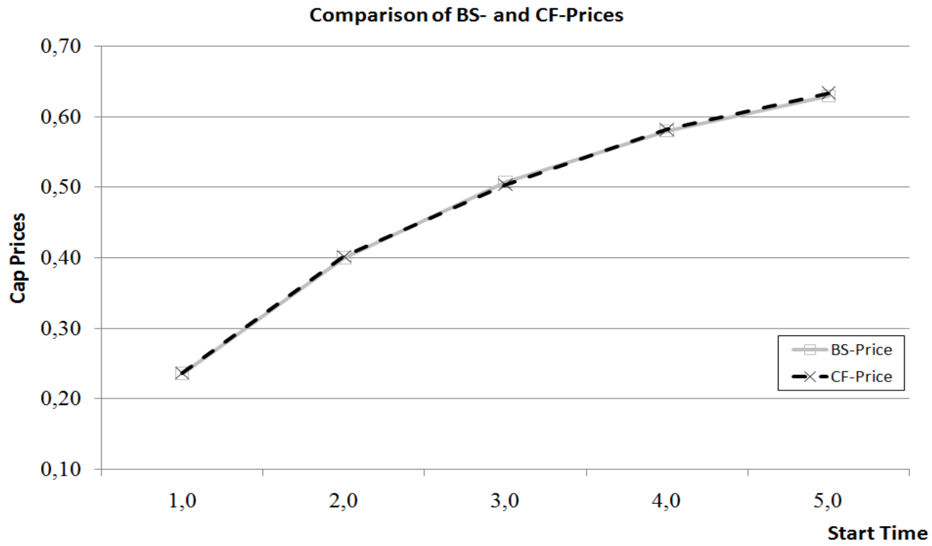

Figure 1: Comparison of the Black-Scholes (BS) prices and the prices computed by characteristic functions (CF). The strike is fixed at 8% (out-of-the-money), the starting time varies between 1 and 5 years and each caps matures one year.

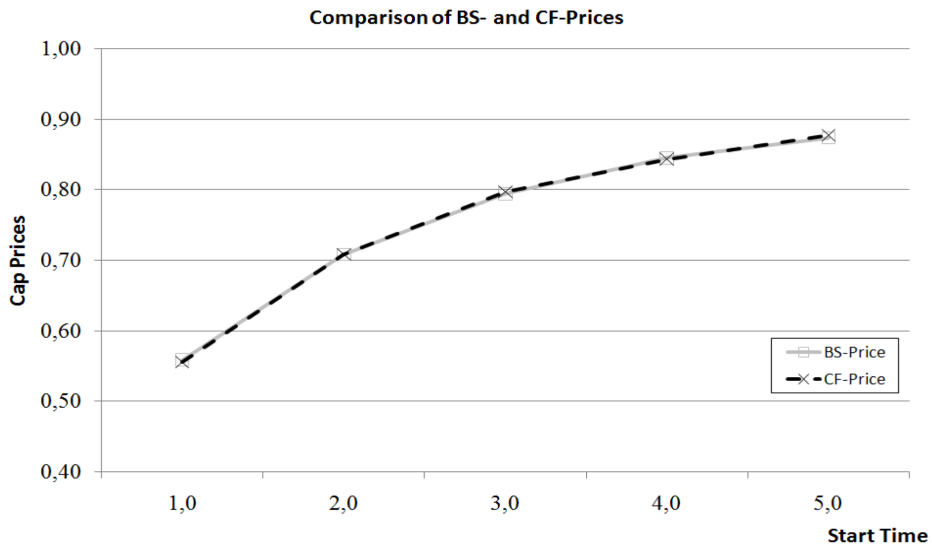

Figure 2: Comparison of the Black-Scholes (BS) prices and the prices computed by characteristic functions (CF). The strike is fixed at 7% (at-the-money), the starting time varies between 1 and 5 years and each caps matures one year.

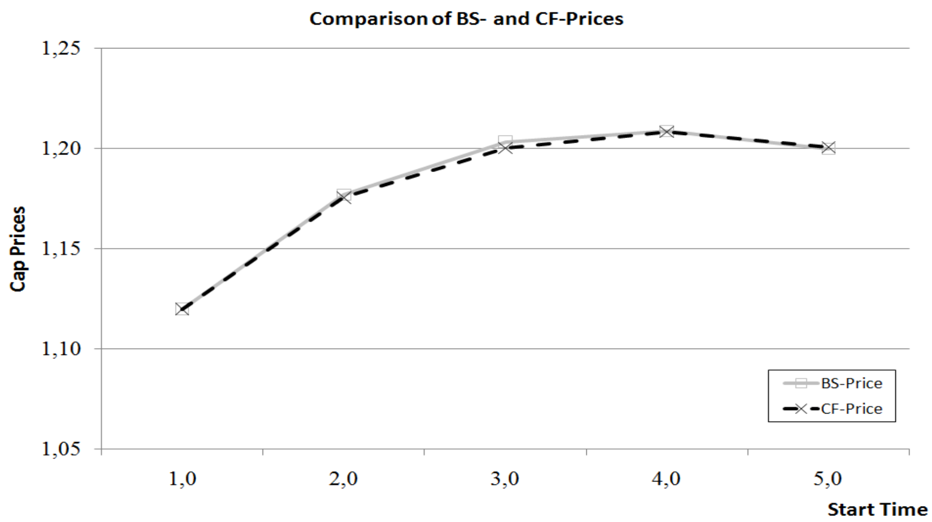

Figure 3: Comparison of the Black-Scholes (BS) prices and the prices computed by characteristic functions (CF). The strike is fixed at 6% (in-of-the-money), the starting time varies between 1 and 5 years and each caps matures one year.

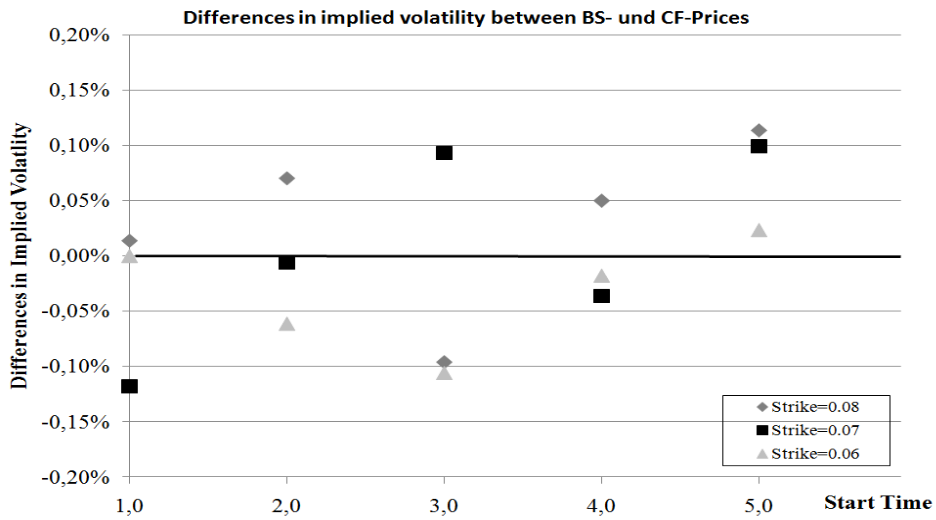

Figure 4: Presentation of the differences in implied volatility between the Black-Scholes and characteristic function prices for caps. The solid line represents caps with strike 8%, the dashed one displays caps with strike 7% and the dotted one brings out the differences in implied volatility for caps with strike 6%. The errors corresponds to the cap prices in Figure 1, Figure 2 and Figure 3.

function delivers slightly higher values. These differences result from numerical imprecisions generated by the multiplication of really high with low values. As described in Section 8 the accuracy of the pricing method increases by increasing the accuracy of the quadrature. Next to the change of strikes we analyzed the behavior of varying implied volatilities. Based on a strike of 8% we used implied volatilities of 10%, 15%, 20%, 25% and 30%. This test incorporates caps starting in 3 years and last 1 year. The results are presented in Table 1, Figure 5 and Figure 6. The error fluctuates between -0.0963% and 0.1015% of implied volatility. The average of the signed differences add up to 0.0048% and the average of the unsigned differences is 0.0561%. Again, one cannot state any trend in the errors as 3 of 5 (60%) prices computed by characteristic functions are higher.

Summarizing, we can observe that the pricing by characteristic functions is conform with the pricing by semi-closed formulas in the one-factor model. We tested the method by varying market situations and did not notice any noticeable systematic problems. Hence the numerical results validate the theoretical analysis.

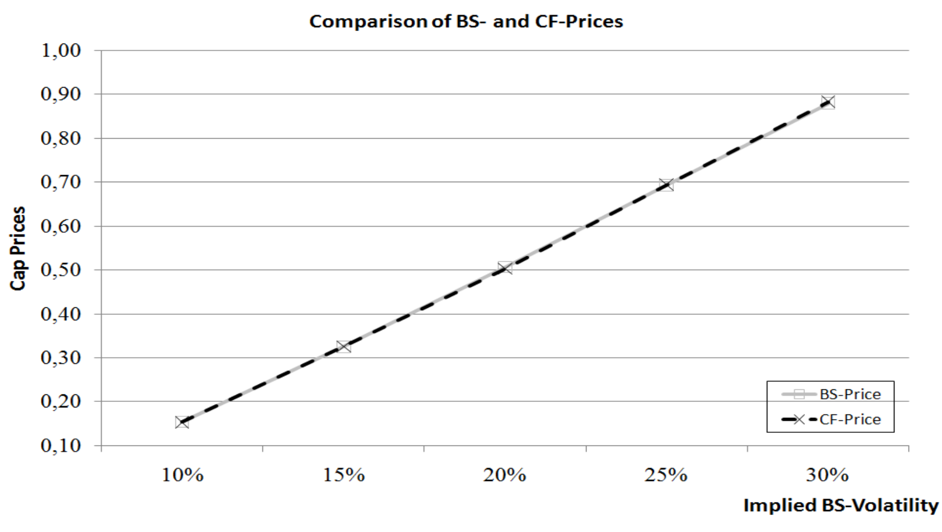

Figure 5: Comparison of the Black-Scholes (BS) prices and the prices computed by characteristic functions (CF). The given implied volatility varies between 0.1 and 0.3. The strike is fixed at 8% (out-of-the-money), the starting time varies between 1 and 5 years and each caps matures one year.

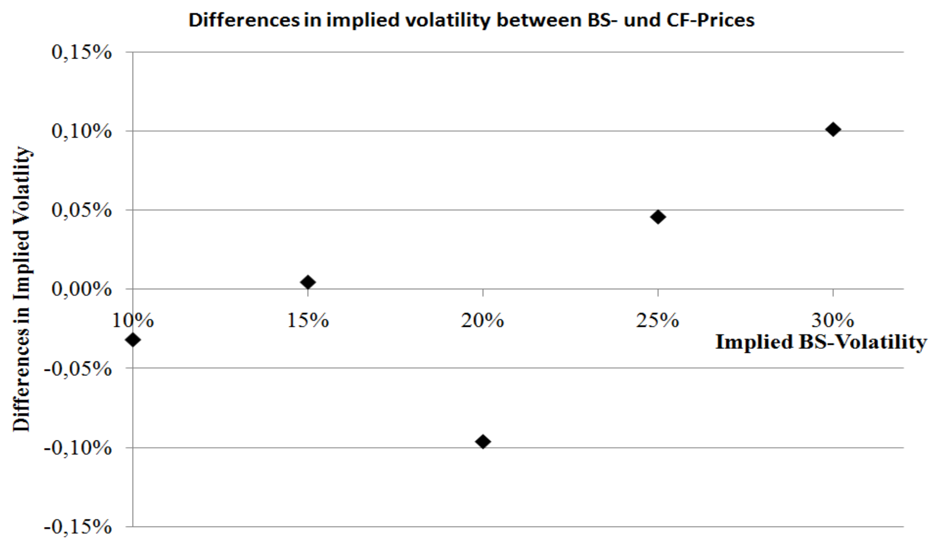

Figure 6: Presentation of the differences in implied volatility between the Black-Scholes and characteristic function prices for caps. The errors corresponds to the cap prices in Figure 5.

8 Numerical Analysis

The pricing of interest rate derivatives by characteristic functions reduces to two main steps. First, the calculation of the model-dependant characteristic function by building up and solving a system of ODEs. Second, the computation of the pricing formulas including an inversion of the characteristic function according to Proposition 7.1. The computation of the characteristic function can be done analytically under some technical conditions as mentioned in Section 7. In contrast, the transform inversion has to be done numerically and the main problem reduces to the computation of an infinite integral of the form

$$\int_0^\infty \frac{\mathrm{Im}\left[\Psi^\chi(a + \imath vb, X_0, 0, T)\exp(-\imath vy)\right]}{v} \mathrm{d}v. \tag{19}$$

First, we investigate the behavior of the integrand close to $v = 0$ and second, we focus on the effect of the numerical integration method.

8.1 Analysis of the Transform Inversion

The integrand has a singularity of order one in $v = 0$, but in the following we will show, that it is removable as the limits $v \to 0$ exists.

Theorem 8.1.
We fix the parameters $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $X_0 = 0 \in \mathbb{R}^n$, $T \in \mathbb{R}$. If the characteristic of the general Cheyette Model $\chi = (H, K, \rho)$ is defined as in Section 5.2, then the limit

$$L = \lim_{v \to 0} \frac{\mathrm{I}\,m[\Psi^{\chi}(a + \imath vb, X_0, 0, T)\exp(-\imath vy)]}{v}$$

exists and is given by

$$L = \exp\Big(\operatorname{Re}[A(0,T,a)]\Big)\Big[\frac{\mathrm{d}}{\mathrm{d}v}\operatorname{Im}[A(0,T,a)] - y\Big].$$

Proof.
The characteristic function is defined by

$$\Psi^{\chi}(u, X_0, 0, T) = \exp[A(0,T,u) + B(0,T,u)X_0]$$

as presented in Section 6.2. Using the initial condition $X_0 = 0$, we obtain

$$\Psi^{\chi}(u, X_0, 0, T) = \exp(A(0,T,u)).$$

$$\begin{aligned} \Rightarrow \operatorname{Im}\Big[\Psi^{\chi}(a + \imath vb, X_0, 0, T)\exp(-\imath vy)\Big] \\ = \operatorname{Im}\Big[\exp(A(0,T,a+\imath vb) - \imath vy)\Big]. \end{aligned}$$

The complex-valued exponential function can be decomposed into real- and imaginary part as exemplarily presented for $w \in \mathbb{C}$

$$\begin{aligned} \exp(w) &= \exp[\operatorname{Re}(w)]\,[\cos(\operatorname{Im}(w)) + \imath\sin(\operatorname{Im}(w))] \\ \Rightarrow \operatorname{Im}\exp(w) &= \exp(\operatorname{Re}(w))\sin(\operatorname{Im}(w)) \end{aligned}$$

$$\Rightarrow \mathrm{Im}[\Psi^{\chi}(a+\imath vb, X_0, 0, T)\exp(-\imath vy)]$$
$$= \exp\Big[\mathrm{Re}(A(0,T,a+\imath vb)-\imath vy)\Big]\sin\Big[\mathrm{Im}(A(0,T,a+\imath vb)-\imath vy)\Big]$$

Thus, the integrand $I(v)$ has the structure

$$\begin{aligned} I(v) =& \frac{1}{v}\exp\Big(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy]\Big) \\ & \sin\Big(\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy]\Big). \end{aligned}$$

In the following we want to apply the rule of L'Hôpital as presented in Appendix A.2. Therefore we have to verify that

(1) $\lim_{v\to 0} \ \exp(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy])\sin(\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy]) = 0,$

(2) $\lim_{v\to 0} \ v = 0.$

The second assumption is trivial and we have to investigate the first one. If we could show, that both conditions are fulfilled, then the limit can be written as

$$\begin{aligned} \lim_{v\to 0} I(v) &= \lim_{v\to 0}\frac{1}{v}\exp\Big(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy]\Big) \\ &\qquad \sin\Big(\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy]\Big) \\ &\overset{\text{L'Hôpital}}{=} \lim_{v\to 0}\frac{1}{\frac{d}{dv}v}\frac{d}{dv}\Big\{\exp\Big(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy]\Big) \\ &\qquad \sin\Big(\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy]\Big)\Big\} \\ &= \lim_{v\to 0}\frac{d}{dv}\Big\{\exp\Big(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy]\Big) \\ &\qquad \sin\Big(\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy]\Big)\Big\}. \end{aligned} \tag{20}$$

The singularity in $v = 0$ would be removed and we could focus on the last equation. But first, we have to verify that

$$\lim_{v\to 0}\{\exp(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy])$$
$$\sin(\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy])\}=0.$$

Therefore we will show

$$\lim_{v\to 0}\mathrm{Im}[A(0,T,a+\imath vb)-\imath vy]=0, \tag{21}$$
$$\lim_{v\to 0}\exp(\mathrm{Re}[A(0,T,a+\imath vb)-\imath vy])=c<\infty, \tag{22}$$

which imply the desired proposition.

First, we concentrate on (21). The function $A(t,T,u)$ is defined by a system of ordinary differential equations (7) and (8). First, we have to solve the ODE (8) for $B(t,T,u)$. In the general Cheyette Model with arbitrary number of factors, the ODE is given by

$$\dot{B}(t) = \rho_1 - K_1^T(t)B(t) \tag{23}$$
$$B(T) = u \tag{24}$$

with fixed parameters $\rho_1\in\mathbb{R}^n$, $u\in\mathbb{C}^n$, $K_1\in\mathbb{R}^{n\times n}$. As presented in Section 5.2, the matrix $K_1\in\mathbb{R}^{n\times n}$ is a diagonal matrix. Thus, the system of ODEs (23) is decoupled and can be solved in every dimension $j=1,...,n$ separately,

$$\dot{B}_j(t)=(\rho_1)_j-(K_1(t))_{jj}(B(t))_j$$
$$\dot{B}_j(T)=u_j=a_j+\imath vb_j.$$

This inhomogeneous ordinary differential equation has a unique solution as exemplarily presented in (Walter 2000)

$$B_j(t)=\exp\Big(-\int_T^t[K_1(s)]_{jj}\mathrm{d}s\Big)$$
$$\left[(a+\imath vb)_j+\int_T^t\left(\rho_{1,j}\exp\left(\int_T^l[K_1(s)]_{jj}ds\right)\right)\mathrm{d}l\right]$$

The coefficient matrix $K_1(t)$ and ρ_1 are real valued thus, the imaginary part of $B_j(t)$ reduces to

$$\operatorname{Im}(B_j(t)) = \exp\Big(-\int_T^t [K_1(s)]_{jj}\mathrm{d}s\Big) v b_j. \tag{25}$$

with

$$\lim_{v\to 0} \operatorname{Im}[B_j(t)] = 0.$$

The function $A(t,T,u)$ is given as a solution to

$$\begin{aligned} \dot{A(t)} &= \rho_0 - K_0 B(t) - \frac{1}{2} B(t)^T H_0 B(t), \\ A(T) &= 0, \end{aligned}$$

with predefined quantities $\rho_0 \in \mathbb{R}$, $B(t) \in \mathbb{C}^n$, $K_0 \in \mathbb{R}^n$ and $H_0 \in \mathbb{R}^n$. The unique solution is given directly via integration

$$\begin{aligned} A(t) &= \int_T^t \rho_0 - K_0(s)B(s) - \frac{1}{2}B(s)^T H_0(s) B(s)\mathrm{d}s \\ &= \int_T^t \rho_0 - \sum_{j=1}^n (K_0)_j B_j(s) - \frac{1}{2}B(s)^T \left[\sum_{j=1}^n (H_0)_{kj} B_j(s)\right] \mathrm{d}s \\ &= \int_T^t \rho_0 - \sum_{j=1}^n (K_0)_j B_j(s) - \frac{1}{2}\sum_{k=1}^n B_k(s) \left[\sum_{j=1}^n (H_0)_{kj} B_j(s)\right]_k \mathrm{d}s. \end{aligned}$$

The complex valued integral can be decomposed in real- and imaginary part. The imaginary part is given by

$$\operatorname{Im}(A(t)) = \int_T^t \operatorname{Im}\left[\rho_0 - K_0(s)B(s) - \frac{1}{2}B(s)^T H_0(s)B(s)\right] \mathrm{d}s$$

and can be divided into three summands:

(1) $\operatorname{Im}(\rho_0) = 0$, as $\rho_0 \in \mathbb{R}$.

(2)

$$\operatorname{Im}[K_0(s)B(s)] = \sum_{j=1}^n [K_0]_j \operatorname{Im}[(B(s))_j]$$

$$= \sum_{j=1}^{n} (K_0)_j \exp(- \int_T^t [K_1(s)]_{jj} ds) v b_j$$
$$\stackrel{v \to 0}{\to} 0$$

(3)

$$\begin{aligned}
&\mathrm{Im}[B(s)^T H_0(s) B(s)] \\
&= \mathrm{Im}\left[\sum_{k=1}^{n} B_k(s) \Big[\sum_{j=1}^{n} (H_0)_{kj} B_j(s) \Big]_k \right] \\
&= \mathrm{Im}\left[\Big\{ \mathrm{Re}\Big(\sum_{k=1}^{n} B_k(s) \Big) + \imath\, \mathrm{Im}\Big(\sum_{k=1}^{n} B_k(s) \Big) \Big\} \right. \\
&\quad \left. \left(\sum_{j=1}^{n} (H_0)_{kj} [\mathrm{Re}(B_j(s)) + \imath\, \mathrm{Im}(B_j(s))] \right) \right] \\
&= \underbrace{\mathrm{Im}\Big[\sum_{k=1}^{n} B_k(s) \Big]}_{\to 0\text{ , for } v \to 0} \left(\sum_{j=1}^{n} (H_0)_{kj} [\mathrm{Re}(B_j(s)) + \imath\, \mathrm{Im}(B_j(s))] \right) \\
&\quad + \mathrm{Re}\Big[\sum_{k=1}^{n} B_k(s) \Big] \sum_{j=1}^{n} (H_0)_k j \underbrace{\mathrm{Im}[B_j(s)]}_{\to 0\text{ , for } v \to 0} \\
&\quad \stackrel{v \to 0}{\to} 0
\end{aligned}$$

This implies

$$\lim_{v \to 0} \mathrm{Im}(A(t, T, 0)) = 0.$$

$$\begin{aligned}
\Rightarrow \lim_{v \to 0} &\mathrm{Im}(A(t, T, 0) - \imath v y) \\
&= \lim_{v \to 0} \mathrm{Im}(A(t, T, 0)) - v y \\
&= 0
\end{aligned}$$

Thus, the first condition (21) is fulfilled. Next, we have to prove condition

(22):

$$\lim_{v\to 0} \exp\left[\mathrm{Re}(A(0,T,a+\imath vb)-\imath vy)\right] = \lim_{v\to 0} \exp\left[\mathrm{Re}(A(0,T,a+\imath vb))\right]$$

The function $\exp\left[\mathrm{Re}(A(0,T,a+\imath vb))\right]$ is continuous with respect to v and thus

$$\lim_{v\to 0} \exp\left[\mathrm{Re}(A(0,T,a+\imath vb))\right] = \exp\left[\mathrm{Re}(A(0,T,a))\right].$$

This function is bounded, if $\mathrm{Re}[A(0,T,a)]$ is bounded. Thus the condition (22) reduces to

$$\mathrm{Re}[A(0,T,a)] = \tilde{c} < \infty.$$

According to previous calculations

$$A(t,T,a) = \int_T^t \rho_0 - \sum_{j=1}^n (K_0)_j B_j(s) - \frac{1}{2}\sum_{k=1}^n B_k(s)\left(\sum_{j=1}^n (H_0)_{kj} B_j(s)\right)_k \mathrm{d}s$$

$$\begin{aligned}\Rightarrow \mathrm{Re}[A(t,T,a)] = \int_T^t & \mathrm{Re}[\rho_0] - \mathrm{Re}\left[\sum_{j=1}^n (K_0)_j B_j(s)\right] \\ & - \frac{1}{2}\mathrm{Re}\left[\sum_{k=1}^n B_k(s)\left(\sum_{j=1}^n [H_0]_{kj} B_j(s)\right)_k\right] \mathrm{d}s\end{aligned}$$

The coefficients ρ_0, K_0, H_0 are fixed and finite. Thus, we have to investigate the real part of $B_j(s)$. If it is bounded, it follows that $\mathrm{Re}[A(0,T,a)]$ is bounded and thus condition (22) is fulfilled,

$$\begin{aligned}B_j(t) = & \exp\left(-\int_T^t (K_1(s))_{jj}\mathrm{d}s\right) \\ & \left[a_j + \int_T^t \{(\rho_1)_j \exp(\int_T^l (K_1(s))_{jj}\mathrm{d}s)\}\mathrm{d}l\right].\end{aligned}$$

The function is real valued if we assume $v = 0$. Again, all coefficients K_1, a

and ρ_1 are fixed and finite. Consequently,

$$\mathrm{Re}[B_j(t)] = B_j(t) = \tilde{c} < \infty.$$

Thus, condition (22) is fulfilled. So far, we have proved Proposition (21) and (22), which were necessary conditions to apply the rule of l'Hôpital. According to (20),

$$\begin{aligned}
\lim_{v\to 0} I(v) &= \lim_{v\to 0} \frac{d}{dv}\Bigg[\exp\Big(\mathrm{Re}(A(0,T,a+\imath vb) - \imath vy)\Big) \\
&\qquad \sin\Big[\mathrm{Im}(A(0,T,a+\imath vb) - \imath vy)\Big]\Bigg] \\
&= \lim_{v\to 0} \frac{d}{dv}\Bigg[\exp\Big(\mathrm{Re}(A(0,T,a+\imath vb))\Big) \\
&\qquad \sin\Big[\mathrm{Im}(A(0,T,a+\imath vb)) - vy\Big]\Bigg] \\
&= \lim_{v\to 0} \Bigg\{\frac{d}{dv}\Big[\exp(\mathrm{Re}(A(0,T,a+\imath vb)))\Big]\sin\Big[\underbrace{\mathrm{Im}(A(0,T,a+\imath vb)) - vy}_{\to 0 \text{ for } v\to 0}\Big] \\
&\qquad + \exp\Big[\mathrm{Re}(A(0,T,a+\imath vb))\Big]\cos\Big[\underbrace{\mathrm{Im}(A(0,T,a+\imath vb)) - vy}_{\to 0 \text{ for } v\to 0}\Big]\Bigg\} \\
&\qquad \Big\{\frac{d}{dv}\mathrm{Im}(A(0,T,a+\imath vb)) - v\Big\} \\
&= \lim_{v\to 0} \underbrace{\exp\Big[\mathrm{Re}(A(0,T,a+\imath vb))\Big]}_{=\tilde{c}<\infty \text{ according to (22)}}\Big(\frac{d}{dv}\mathrm{Im}(A(0,T,a+\imath vb)) - y\Big).
\end{aligned}$$

Finally, we have to show that $\lim\limits_{v\to 0}\frac{d}{dv}\mathrm{Im}(A(0,T,a+\imath vb))$ is bounded. As already shown, the imaginary part of $A(0,T,a+\imath vb)$ can be written as

$$\begin{aligned}
\mathrm{Im}[A(0,T,a+\imath vb)] = \int_T^0 &\mathrm{Im}(\rho_0) - \mathrm{Im}\Big(K_0(s)B(s)\Big) \\
&- \frac{1}{2}\mathrm{Im}\Big(B(s)^T H_0(s)B(s)\Big)\mathrm{d}s.
\end{aligned}$$

The coefficients ρ_0, K_0, H_0 are bounded and independent of v. Thus, the derivation with respect to v just influences $B(s)$. If we can show, that

$$\lim_{v\to 0} \frac{d}{dv} \operatorname{Im}[B(s,T,a+\imath vb)] = c_1 < \infty,$$

holds, that would imply

$$\frac{d}{dv} \operatorname{Im}[A(0,T,a+\imath vb)] = c_2 < \infty.$$

The boundedness of this expression completes the proof. As shown in (25)

$$\operatorname{Im}\Big(B_j(s)\Big) = \exp\Big(-\int_T^s (K_1(l))_{jj}\mathrm{d}l\Big)vb_j.$$

$$\Rightarrow \frac{d}{dv}\operatorname{Im}\Big(B_j(s)\Big) = \exp\Big(-\int_T^s [K_1(l)]_{jj}\mathrm{d}l\Big)b_j$$

The coefficients K_1 and b_j are fixed and finite, thus $\frac{d}{dv_j}\operatorname{Im} B_j(s)$ is bounded, which completes the proof concerning the existence of the limit.
The limit is given by

$$\begin{aligned} L &= \lim_{v\to 0} I(v) \\ &= \lim_{v\to 0} \exp\Big(\operatorname{Re}(A(0,T,a+\imath vb))\Big)\Big[\frac{d}{dv}\operatorname{Im}(A(0,T,a+\imath vb)) - y\Big] \\ &= \exp\Big(\operatorname{Re}(A(0,T,a))\Big)\Big[\frac{d}{dv}\operatorname{Im}(A(0,T,a)) - y\Big] \end{aligned}$$

□

Theorem 8.2.
We fix the parameters $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $X_0 = 0 \in \mathbb{R}^n$, $T \in \mathbb{R}$. The characteristic $\chi = (H, K, \rho)$ representing the Ho-Lee Model is defined as in Section 5.2, then the limit

$$L = \lim_{v \to 0} \frac{\mathrm{I}\, m[\Psi^{\chi}(a + \imath v b, X_0, 0, T) \exp(-\imath v y)]}{v}$$

exists and is given by

$$L = \exp\Big[- fT + \frac{c^2 T}{6}(-2T^2 + 3(Ta + a^2))\Big] \Big[\frac{c^2 T}{6}(3Tb + 2ab) - y\Big].$$

Proof.
In the Ho-Lee Model, the function $A(0, T, a + \imath v b)$ is defined by

$$A(0, T, a + \imath v b) = - fT + \frac{c^2 T}{2}\Big[- 2T^2 + 3T(a + \imath v b) + 3(a + \imath v b)^2\Big]$$

Thus, the real and imaginary parts are given by

$$\mathrm{Re}(A(0, T, a + \imath v b)) = \quad -fT + \tfrac{c^2 T}{6}[-2T^2 + 3(Ta + a^2 - v^2 b^2)]$$

and

$$\mathrm{Im}(A(0, T, a + \imath v b)) = \quad \tfrac{c^2 T}{6}[3Tvb + 2avb]$$

$$\begin{aligned} \Rightarrow \frac{d}{dv}&\{ \mathrm{Im}\, A(0, T, a + \imath v b)\} \\ &= \frac{c^2 T}{6}[3Tb + 2ab]. \end{aligned}$$

This implies

$$\begin{aligned} L &= \lim_{v \to 0} \exp\Big(\mathrm{Re}(A(0, T, a + \imath v b))\Big) \Big[\frac{d}{dv} \mathrm{Im}(A(0, T, a + \imath v b) - y)\Big] \\ &= \exp\Big[- fT + \frac{c^2 T}{6}(-2T^2 + 3(Ta + a^2))\Big]\Big[\frac{c^2 T}{6}(3Tb + 2ab) - y\Big]. \end{aligned}$$

□

In addition to the analytical proof of the existence of the limit, we tested the behavior of the integrand close to zero numerically. The shape of the integrand is determined by the model, the parameters $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $X_0 \in \mathbb{R}^n$ and $T \in \mathbb{R}_+$. We tested numerous parameter sets and different (one-factor) models ($n = 1$) to understand the behavior close to zero. Mainly we identified two types of function shapes just depending on the parameter $y \in \mathbb{R}$. If y is positive, the function is negative and strictly increasing to 0 and if y is negative, the function has positive values and is strictly decreasing to 0. Exemplarily we plotted two integrand functions (Test case 5 and 9) in the interval $v \in [10^{-10}, 300]$ with step size $h = 10^{-6}$ and the functions are shown in Figure 7.

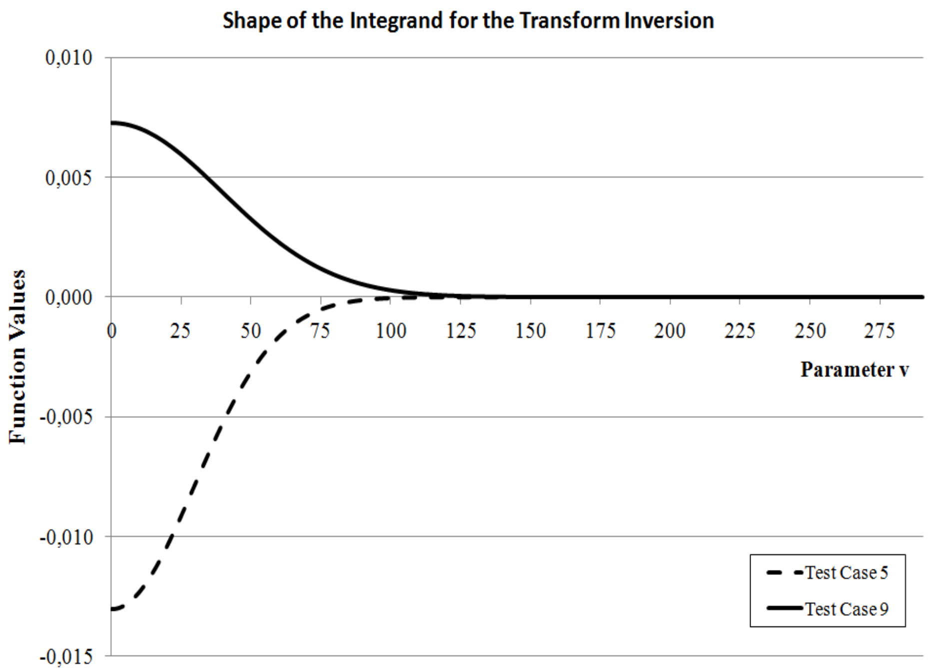

Figure 7: Shape of the integrand of the transform inversion for two different parameter sets in the interval $[10^{-10}, 300]$ and step size $h = 10^{-6}$. Test Case 5: $a = -6$, $b = 1$, $y = 0.018707283$, $x = 0$, $T = 5$, $c = 0.02$, $f = 0.06$; Test Case 9: $a = -4$, $b = 1$, $y = -008943557$, $x = 0$, $T = 3$, $c = 0.02$, $f = 0.06$.

Furthermore, we concentrated on the function behavior in a small neighborhood of zero. Therefore, we evaluated the integrand in the interval $[10^{-14}, 10^{-6}]$ with a step size of $h = 10^{-12}$ and plotted the results in Figure 8.

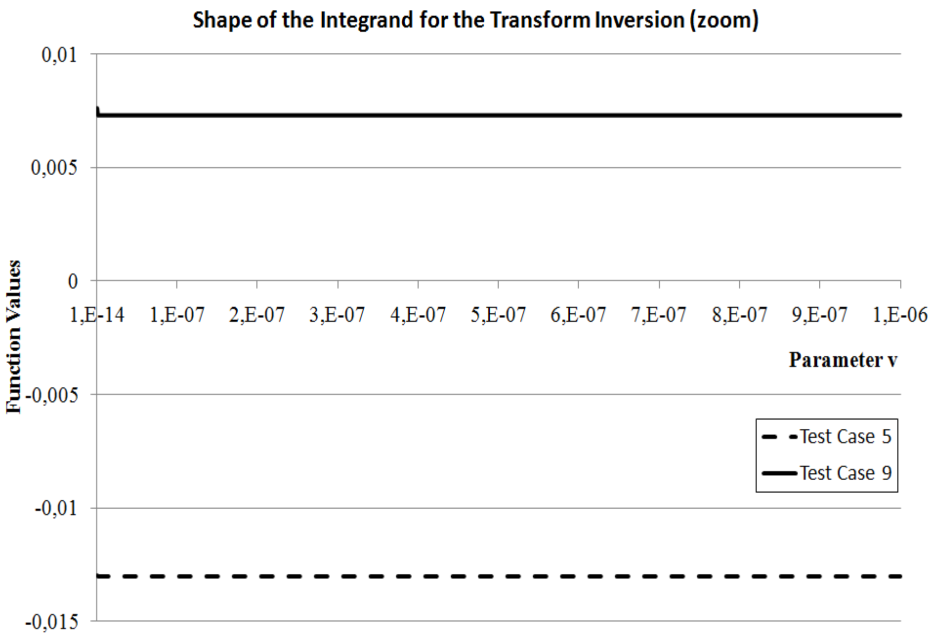

Figure 8: Shape of the integrand of the transform inversion for two different parameter sets in the interval $[10^{-14}, 10^{-6}]$ and step size $h = 10^{-12}$. Test Case 5: $a = -6$, $b = 1$, $y = 0.018707283$, $x = 0$, $T = 5$, $c = 0.02$, $f = 0.06$; Test Case 9: $a = -4$, $b = 1$, $y = -008943557$, $x = 0$, $T = 3$, $c = 0.02$, $f = 0.06$.

The empirical results confirm the existence of the limit $v \to 0$ and the values of the limit corresponds to the theoretical values.

Parameter set	Theoretical Value $\lim\limits_{v \to 0}$	Function Value at $v = 10^{-14}$	Absolute Error
Test Case 5	-0.012978697590	-0.012978705919	$8.3296\ 10^{-9}$
Test Case 9	0.007628181395	0.007628181293	$1.0167\ 10^{-10}$

Table 2: Presentation of the values of the integrand close to zero and comparison to the theoretical results of the limit. We focus on two cases specified by the following parameters: Test Case 5: $a = -6$, $b = 1$, $y = 0.014975$, $x = 0$, $T = 5$, $c = 0.0207$; Test Case 9: $a = -4$, $b = 1$, $y = -0.001053$, $x = 0$, $T = 3$, $c = 0.03501$.

Additional to the behavior of the integrand close to zero, we investigated the properties of the integrand function in the limit as $v \to \infty$. The integrand tends to zero as v tends to infinity, as already indicated in Figure 7. In most of the cases it is sufficient to incorporate parameter values up to 600, because the absolute function value decreases under the level of 10^{-16} and does not increase afterwards.

8.2 Effect of the Numerical Integration Method

The shape of the function changes mainly in dependance on the model and the evaluation point. Therefore, we analyzed the influence of the quadrature method on the prices. All in all, we incorporated the Simpson, Gauss-Legendre, Gauss-Laguerre and adjusted Gauss-Laguerre quadrature in the analysis.
The Simpson quadrature is one of the easiest and most robust numerical integration method. The supporting points are equidistantly distributed and the accuracy depends on the grid size with a power of 4. The quadrature methods of Gauss has more approximation power and thus, they deliver better results in less time. In contrast to the basic quadratures, the supporting points are not distributed equidistantly. The Gauss quadrature incorporating n-points is constructed to yield exact results for polynomials of degree $2n-1$. Generally, the Gauss quadratures approximate

$$\int_{x_1}^{x_2} f(x)dx \approx \sum_{j=0}^{n-1} \omega(x_j) f(x_j).$$

The choice of abscissas x_j and weights $\omega(x_j)$ characterize the different methods of Gauss quadrature (Press 2002). Given some orthonormal set of polynomials, the abscissas turn out to be the distinct roots of them. The orthonormality condition is constructed with respect to a given weight function $\omega(x)$ and is defined by the scalar product

$$< f, g >= \int_a^b \omega(x) f(x) g(x) \mathrm{d}x.$$

We focussed mainly on two traditional Gauss quadrature rules, Gauss - Legendre and Gauss-Laguerre. The Gauss-Legendre method is based on the weight function

$$\omega(x) = 1, \text{ for } -1 < x < 1,$$

and the Legendre polynomials P_j are defined recursively by

$$(j+1)P_{j+1} = (2j+1)xP_j - jP_{j-1}.$$

The Gauss-Legendre quadrature approximates finite integrals and thus, we have to estimate a reasonable integration limit first. We fixed the limit of integration, when the absolute value of the integrand falls below 10^{-16}. Numerical results have shown, that the typical integration limit is about 600. Furthermore, the integrand of the transform inversion defined in Proposition 7.1 tends monotonically to zero, thus it is not possible, that the absolute function value will increase outside of the integration area.

The Gauss-Laguerre method uses the weight function

$$w(x) = x^{\alpha} \exp(-x), \text{ for } 0 < x < \infty,$$

and the Laguerre polynomials L_{j+1}^{α} are defined recursively by

$$(j+1)L_{j+1}^{\alpha} = (-x + 2j + \alpha + 1)L_j^{\alpha} - (j+\alpha)L_{j-1}^{\alpha}.$$

This quadrature rule can directly be used to approximate infinite integrals $\int_0^{\infty} f(x)\mathrm{d}x$. Furthermore, we investigated the use of an adjustment of the Gauss-Laguerre method suggested by R. Sagar et al. (Sagar, Schmider & Smith 1992). In their paper, they brought out that a simple substitution in the quadrature rule of Gauss-Laguerre increases the accuracy especially for Fourier-Transforms. They assume an exponential structure of the integrand function and illustrate their results in a chemical application computing the atomic form factor for neon. In general, Sagar et al. want to compute the abstract integral

$$F(k) = \int_0^{\infty} \omega(x) f(x) \exp(-\imath kx)\mathrm{d}x$$

with weight function

$$\omega(x) = x^m \exp(-\alpha x).$$

The aim is the application of the Gauss-Laguerre quadrature constructed by the given weight function, which is a generalization of the previously discussed one. First, we substitute

$$x = \varphi(z) = \frac{z}{\alpha + \imath k}, \tag{26}$$

which implies

$$\begin{aligned}
F(k) &= \int_0^\infty \varphi(z)^m \exp(-\alpha\varphi(z)) f(\varphi(z)) \exp(-\imath k \varphi(z)) \varphi'(z) \mathrm{d}z \\
&= \int_0^\infty \varphi'(z)\varphi(z)^m \exp(-\varphi(z)[\alpha + \imath k]) f(\varphi(z)) \mathrm{d}z \\
&= \frac{1}{\alpha + \imath k} \int_0^\infty \left(\frac{z}{\alpha + \imath k}\right)^m \exp(-z) f(\frac{z}{\alpha + \imath k}) \mathrm{d}z \\
&= \left(\frac{1}{\alpha + \imath k}\right)^{m+1} \int_0^\infty z^m \exp(-z) f(\frac{z}{\alpha + \imath k}) \mathrm{d}z \\
&\approx \left(\frac{1}{\alpha + \imath k}\right)^{m+1} \sum_{j=1}^n f(\frac{z_j}{\alpha + \imath k}) \omega(z_j).
\end{aligned}$$

This numerical integration algorithm can be applied to compute the transform inversion of Theorem 7.1. Therefore, we have to compute the integral

$$\begin{aligned}
I(y) &= \int_0^\infty \frac{\mathrm{Im}\left[\Psi^\chi(a + \imath v b, X_0, 0, T) \exp(-\imath v y)\right]}{v} \mathrm{d}v \\
&= \int_0^\infty \mathrm{Im}\left[\frac{\Psi^\chi(a + \imath v b, X_0, 0, T) \exp(-\imath v y)}{v}\right] \mathrm{d}v \\
&= \mathrm{Im}\left[\int_0^\infty \frac{\Psi^\chi(a + \imath v b, X_0, 0, T) \exp(-\imath v y)}{v} \mathrm{d}v\right].
\end{aligned}$$

Now, we have to apply the general form to

$$f(x) = \frac{\Psi^\chi(a + \imath x b, X_0, 0, T)}{x} x^{-m} \exp(\alpha x).$$

$$
\begin{aligned}
\Rightarrow I(y) &= \mathrm{Im}\Big[\int_0^\infty f(v)\omega(v)\exp(-\imath vy)\mathrm{d}v\Big] \\
&\approx \mathrm{Im}\Big[\Big(\frac{1}{\alpha+\imath y}\Big)^{m+1}\sum_{j=1}^{n} f(\frac{z_j}{\alpha+\imath y})\omega(z_j)\Big] \\
&= \mathrm{Im}\Big[\Big(\frac{1}{\alpha+\imath y}\Big)^{m+1}\sum_{j=1}^{n}\frac{\Psi^\chi(a+\imath b\frac{z_j}{\alpha+\imath y},X_0,0,T)}{z_j}(\alpha+\imath y) \\
&\quad \Big(\frac{z_j}{\alpha+\imath y}\Big)^{-m}\exp(\frac{\alpha z_j}{\alpha+\imath y})\omega(z_j)\Big] \\
&= \mathrm{Im}\Big[\Big(\frac{1}{\alpha+\imath y}\Big)^{m+1}\sum_{j=1}^{n}\Psi^\chi(a+\imath b\frac{z_j}{\alpha+\imath y},X_0,0,T) \\
&\quad \Big(\frac{\alpha+\imath y}{z_j}\Big)^{m+1}\exp(\frac{\alpha z_j}{\alpha+\imath y})\omega(z_j)\Big] \\
&= \mathrm{Im}\Big[\sum_{j=1}^{n}\Psi^\chi(a+\imath b\frac{z_j}{\alpha+\imath y},X_0,0,T)\frac{\exp(\frac{\alpha z_j}{\alpha+\imath y})}{z_j^{m+1}}\omega(z_j)\Big] \\
&= \sum_{j=1}^{n}\mathrm{Im}\Big[\Psi^\chi(a+\imath b\frac{z_j}{\alpha+\imath y},X_0,0,T)\frac{\exp(\frac{\alpha z_j}{\alpha+\imath y})}{z_j^{m+1}}\omega(z_j)\Big]
\end{aligned}
$$

The presented algorithm is a generalization of the well-known Gauss-Laguerre quadrature and the weight function depends on two parameters $\alpha \in \mathbb{R}$ and $m \in \mathbb{R}$, $m > -1$. The results in (Sagar et al. 1992) demonstrate a benefit on the computation of Fourier-Transforms in dependance of the choice of α and m. If one chooses $\alpha = 1$, the new method and the normal one correspond. Sagar et al. promote a choice of $\alpha = 8$ and $m = 1$ in their paper to achieve the best results.

We tested the quadrature rules by pricing caps and compare the results in terms of implied volatility as presented in Section 7.3. In the following we will demonstrate some results exemplarily for caps with strike 6%, starting in one year, maturing in one year and an implied volatility of 20%. The price in the Black-Scholes model with a nominal of 100 equals 1.1198. We investigated in particular in the convergence of the quadrature, the stability and the speed. First, we focus on the Simpson quadrature whose precision is determined by the fractional accuracy given by the parameter EPS. Decreasing EPS should imply increasing precision and the results in Table 3 demonstrate the convergence.

EPS	CF - Price	Diff. in implied volatility
10^{-5}	1.120001	0.01236%
10^{-6}	1.119960	0.00978%
10^{-7}	1.119955	0.00941%
10^{-8}	1.119955	0.00941%
10^{-9}	1.119954	0.00936%

Table 3: Presentation of the results of cap pricing with varying accuracy by using Simpson quadrature.

Second, we tested the Gauss-Legendre quadrature and therefore we limited the infinite integration to the interval $[0, 524]$ as already explained. The number of points n controls the approximation power of this method. Table 4 shows no error reduction by increasing the number of points, but the results are already accurate for $n = 50$.

n	CF - Price	Diff. in implied volatility
50	1.11995508	0.0094127%
100	1.11995508	0.0094127%
150	1.11995508	0.0094127%

Table 4: Presentation of the results of cap pricing with increasing number of supporting points in the Gauss-Legendre quadrature.

Third, we analyzed the Gauss-Laguerre quadrature by varying the number of supporting points n and the parameter α determining the weight function $\omega(x)$. Thereby we incorporated 10 different caps in the analysis. Table 6 shows a huge influence of α on the accuracy of the quadrature and as well on the price for a single cap. Nevertheless, we can observe the convergence of the prices by increasing number of points n for any parameter α. Summarizing, we achieve the best results for $\alpha = 2$ and $n = 150$. Last, we analyzed the behavior of the adjusted Gauss-Laguerre method proposed by R. Sagar. We tested several combinations of parameters α and m and detected a large influence. The analysis incorporated 10 caps and we present the results exemplarily for one cap in Table 7. In several cases, the parameters were in fact worse, because the resulting price could no longer be inverted. Consequently we state, that this method is not stable in our application, although it seems to converge.

Furthermore, we clocked the CPU time[2] of the algorithm to compute the price of one cap with reasonable accuracy, see Table 5.

Quadrature rule	Accuracy	CPU time in sec.
Simpson	10^{-12}	4.7420
Gauss-Legendre	10^{-14}	0.00162
Gauss-Laguerre	10^{-14}	0.00218

Table 5: Comparison of the quadrature methods with respect to the spent CPU time to compute one cap.

These results show clearly, that the Gauss quadrature is superior to the Simpson method. Summarizing the influences of the quadrature, we see that the best results where obtained by Gauss-quadratures. Especially the Laguerre method with $\alpha = 2$ provides reliable results in short time.

9 Calibration

In order to use an interest rate model in practice it needs to be calibrated to liquidly traded interest rate options. The calibration of one-factor Cheyette Models has been investigated in (Beyna & Wystup 2010). The analysis was based on semi-closed formulas (Henrard 2003) existing only for one-factor models. Thereby it was shown, that the optimization problem owns several local minima and the optimization method influences the accuracy. The method 'Simulated Annealing' delivers the best and most reliable results. The calibration of multi-factor models can be performed by using characteristic functions for the valuation. Thereby we focus on the calibration to caps and floors. The pricing of caps by characteristic functions is shown in Section 7 and can be performed quickly as shown in Table 5. The complexity of the price computation is almost independent of the number of factors included in the model, because the pricing formula just includes the numerical computation of a one-dimensional integral as presented in Section 7. The

[2]We used a Windows based PC with Intel Core 2 Duo CPU @ 1.66 GHz and 3.25 GB RAM.

calibration problem is constructed as a (global) minimization of the squared differences in implied volatility

$$E = \min_{\theta \in \Theta} \sum_{caps} |\sigma_{impl} - \sigma_{impl}^{Ch}(\theta)|^2, \tag{27}$$

where $\Theta \subset \mathbb{R}^n$ denotes the set of all parameter sets, σ_{impl} names the implied volatility observed at the market and $\sigma_{impl}^{Ch}(\theta)$ identifies the implied (Black-Scholes) volatility as computed by using the parameter set θ. The computation of the implied volatility $\sigma_{impl}^{Ch}(\theta)$ is divided into the computation of the price in the Cheyette Model by using characteristic functions and the parameter set ν and the inversion of the price concerning the implied (Black-Scholes) volatility.

The computation of the solution to the minimization problem can be performed by the Simulated Annealing algorithm. The method does not guarantee locating the global minimum, but reaches it with high probability as presented for the one-factor model in (Beyna & Wystup 2010).

The fundamental goal of the calibration is to determine the parameters θ^{min}, that reproduce the current market state best. The parameters θ^{min} fully specify the interest rate model and afterwards we can use it to price exotic interest rate products like snowballs or Bermudan swaptions. Therefore, one can use the valuation by Partial Differential Equations or Monte Carlo Simulation.

10 Conclusion

The use of Fourier Transforms for valuing interest rate derivatives forms a very powerful technique. The computation of the expected value of the final payoff simplifies by exploiting the probability density function of the model dynamic. In particular, the necessary integration becomes independent of the dimension of the state variables. The classification of the Cheyette Model dynamic as an affine-diffusion process allows us to apply characteristic functions. Thereby we assume an exponential structure as suggested by (Duffie et al. 1999) and specify the characteristic function via two coefficient functions. These functions are given by (unique) solutions to a system of complex-valued ordinary differential equations (Riccati equation). The general structure of Cheyette Models enables us to solve these ODEs analytically for an arbitrary number of factors incorporated in the model. Thus the framework is valid for any multi-factor model in the class of Cheyette Models.

The general setup provides formulas for pricing interest rate derivatives, in particular, options. If the characteristic function is known explicitly, the computation of the price can essentially be reduced to a one-dimensional integral. The analysis of the integrand verifies that the integration is numerically stable, because a singularity can be removed as presented in Section 8.

The theoretical framework is confirmed by some numerical tests of pricing caps in the (one-factor) Ho-Lee Model. There exist semi-closed formulas for one-factor models only and we compared the prices to the ones obtained by the characteristic function method. After showing the consistency for one-factor models empirically, we assume, that the extension to multi-factor models is valid as well. Thus, we applied this pricing technique to calibrate multi-factor models to caps representing the current state of the market.

Summarizing, we showed that the Fourier Transform technique is applicable to Cheyette Models. This method is powerful as it is fast and almost independent of the number of model factors.

A Appendix

A.1 Convergence of the Gauss-Laguerre quadrature

α	n	CF - Price	Diff. in implied volatility
-0.5	50	1.229825	0.2055%
-0.5	100	1.119972	0.0105%
-0.5	150	1.119966	0.0101%
0.0	50	1.122884	0.1991%
0.0	100	1.119955	0.0094%
0.0	150	1.119955	0.0094%
0.5	50	1.122752	0.1906%
0.5	100	1.119932	0.0079%
0.5	150	1.119921	0.0072%
1.0	50	1.122592	0.1803%
1.0	100	1.119900	0.0058%
1.0	150	1.119873	0.0040%
2.0	50	1.122197	0.1548%
2.0	100	1.119737	0.0047%
2.0	150	1.119809	$6.3\ 10^{-5}$%
3.0	50	1.121711	0.1233%
3.0	100	1.119553	0.0167%
3.0	150	1.119685	0.0081%
4.0	50	1.121139	0.0863%
4.0	100	1.119322	0.0317%
4.0	150	1.119530	0.0182%
5.0	50	1.119048	0.0496%
5.0	100	1.120488	0.0441%
5.0	150	1.119344	0.0303%

Table 6: Results of cap pricing with characteristic functions by using the Gauss-Laguerre quadrature with varying parameters.

m	α	n	CF - Price	Diff. in implied volatility
1.0	2.0	50	1.070708	3.4532%
1.0	2.0	100	1.118134	0.1091%
1.0	2.0	150	1.120549	0.0480%
1.0	1.0	50	1.118542	0.0826%
1.0	1.0	100	1.116211	0.2350%
1.0	1.0	150	1.116238	0.2332%
1.0	3.0	50	0.964673	–
1.0	3.0	100	1.114117	0.2350%
1.0	3.0	150	1.121178	0.0888%
1.0	8.0	50	0.714511	–
1.0	8.0	100	0.883396	–
1.0	8.0	150	1.004729	11.41%
1.0	4.0	50	0.879689	–
1.0	4.0	100	1.074277	–
1.0	4.0	150	1.120993	0.0768%
1.0	5.0	50	0.818515	–
1.0	5.0	100	1.020851	8.2575%
1.0	5.0	150	1.104981	0.9832%
2.0	2.0	50	1.072631	3.3048%
2.0	2.0	100	1.118087	0.1123%
2.0	2.0	150	1.120424	0.0399%
2.0	3.0	50	0.967571	–
2.0	3.0	100	1.114451	0.3508%
2.0	3.0	150	1.121109	0.0843%
2.0	4.0	50	0.882527	–
2.0	4.0	100	1.075258	3.1044%
2.0	4.0	150	1.121072	0.0819%

Table 7: Results of the cap pricing with characteristic functions by using the adjusted Gauss-Laguerre quadrature with varying parameters. '–' denotes, that the CF-price could not be inverted reasonably with respect to the implied volatility.

A.2 Mathematical Background

Proof of the transform inversion presented in Proposition 7.1.

Proof.
The proof is based on the ideas of Duffie et al. (Duffie et al. 1999), but it contains some necessary adjustments.
For $0 < \tau < \infty$ and a fixed $y \in \mathbb{R}$,

$$
\begin{aligned}
&\frac{1}{2\pi}\int\limits_{-\tau}^{\tau} \frac{\exp(\imath v y)\Psi^{\chi}(a-\imath v b, X, 0, T) - \exp(-\imath v y)\Psi^{\chi}(a+\imath v b, X, 0, T)}{\imath v}\mathrm{d}v \\
&\quad = \frac{1}{2\pi}\int\limits_{-\tau}^{\tau}\left(\int\limits_{\mathbb{R}} \frac{\exp[-\imath v(z-y)] - \exp[\imath v(z-y)]}{\imath v}\mathrm{d}G_{a,b}(z;x,T,\chi)\right)\mathrm{d}v \\
&\quad \overset{\text{Fubini}}{=} \frac{-1}{2\pi}\int\limits_{\mathbb{R}}\left(\int\limits_{-\tau}^{\tau} \frac{\exp[-\imath v(z-y)] - \exp[\imath v(z-y)]}{\imath v}\mathrm{d}v\right)\mathrm{d}G_{a,b}(z;x,T,\chi).
\end{aligned}
$$

The theorem of Fubini is applicable, because

$$
\lim_{y\to\infty} G_{a,b}(y,x,T,\chi) = \Psi^{\chi}(a,x,0,T) < \infty
$$

and

$$
|\exp(\imath v) - \exp(\imath u)| \leq |v-u| \;, \forall u,v \in \mathbb{R}.
$$

Next we note that for $\tau > 0$,

$$
\begin{aligned}
&\int\limits_{-\tau}^{\tau} \frac{\exp[-\imath v(z-y)] - \exp[\imath v(z-y)]}{\imath v}\mathrm{d}v \\
&= \int\limits_{-\tau}^{\tau} \frac{\cos[v(z-y)] - \cos[v(z-y)] - 2\imath\sin[v(z-y)]}{\imath v}\mathrm{d}v \\
&= \int\limits_{-\tau}^{\tau} \frac{-2}{v}\sin[v(z-y)]\mathrm{d}v \\
&= -2\,\mathrm{sgn}(z-y)\int\limits_{-\tau}^{\tau} \frac{\sin(v|z-y|)}{v}\mathrm{d}v
\end{aligned}
$$

is bounded simultaneously in z and τ, for each fixed y. Thereby, we define

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

The Bounded Convergence Theorem implies

$$\begin{aligned}
&\lim_{\tau\to\infty} \frac{1}{2\pi} \int_{-\tau}^{\tau} \frac{\exp[\imath vy]\Psi^{\chi}(a-\imath vb, x, 0, T) - \exp[-\imath vy]\Psi^{\chi}(a+\imath vb, x, 0, T)}{\imath v} \mathrm{d}v \\
&= \lim_{\tau\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \underbrace{\left(\int_{-\tau}^{\tau} \frac{\exp[-\imath v(z-y)] - \exp[\imath v(z-y)]}{\imath v} \mathrm{d}v \right)}_{=-2\operatorname{sgn}(z-y) \int_{-\tau}^{\tau} \frac{\sin[v|z-y|]}{v} \mathrm{d}v} \mathrm{d}G_{a,b}(z; x, T, \chi) \\
&= \lim_{\tau\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} -2\operatorname{sgn}(z-y) \underbrace{\left(\int_{-\tau}^{\tau} \frac{\sin[v|z-y|]}{v} \mathrm{d}v \right)}_{\to\pi \text{ for } \tau\to\infty} \mathrm{d}G_{a,b}(z; x, T, \chi) \\
&= \frac{-2}{2\pi} \int_{\mathbb{R}} \operatorname{sgn}(z-y)\pi \mathrm{d}G_{a,b}(z; x, T, \chi) \\
&= -\int_{\mathbb{R}} \operatorname{sgn}(z-y) \mathrm{d}G_{a,b}(z; x, T, \chi) \\
&= -\Psi^{\chi}(a, x, 0, T) + \left(G_{a,b}(y, x, T, \chi) + G_{a,b}(y^-, x, T, \chi) \right),
\end{aligned}$$

where $G_{a,b}(y^-; x, T, \chi) = \lim_{z\to y, z\leq y} G_{a,b}(z, x, T, \chi)$. The integrability of the characteristic function (assumption in the proposition) in combination with the dominated convergence implies

$$\begin{aligned}
G_{a,b}(y, x, T, \chi) =& \frac{\Psi^{\chi}(a, X_0, 0, T)}{2} \\
&+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\imath v} \Big\{ \exp[\imath vy]\Psi^{\chi}(a-\imath vb, X_0, 0, T) \\
&- \exp[-\imath vy]\Psi^{\chi}(a+\imath vb, X, 0, T) \Big\} \mathrm{d}v
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\Psi^{\chi}(a, X_0, 0, T)}{2} \\
&-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\Psi^{\chi}(a+\imath vb, X_0, 0, T)\exp(-\imath vy)\right]}{v}\mathrm{d}v,
\end{aligned}
$$

where we use the fact that $\Psi^{\chi}(a-\imath vb, X_0, 0, T)$ is the complex conjugate of $\Psi^{\chi}(a+\imath vb, X_0, 0, T)$.

$\square$

Theorem A.1 (Uniqueness and Existence Theorem).
Suppose that $b : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and $B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{m\times n}$ are continuous and satisfy the following conditions:

(a) $|b(x, t) - b(\hat{x}, t)| \leq L|x - \hat{x}|$,
$|B(x, t) - B(\hat{x}, t)| \leq L|x - \hat{x}|$, *for all* $0 \leq t \leq T, x, \hat{x} \in \mathbb{R}^n$

(b) $|b(x, t)| \leq L(1 + |x|)$
$|B(x, t)| \leq L(1 + |x|)$, *for all* $0 \leq t \leq T, x, \hat{x} \in \mathbb{R}^n$

for some constant L. Let X_0 be any $\mathbb{R}^n$-valued random variable such that

(c) $E[|X_0|^2] < \infty$

(d) X_0 *is independent of* $\mathcal{W}^+(0)$, *where the* σ*-algebra* $\mathcal{W}^+(t) = \sigma\Big(W(s) - W(t)|s \geq t\Big)$ *is the future of the* m*-dimensional Brownian Motion* W *beyond time* t*.*

Then there exists a unique solution $X \in L^2(\mathbb{R}^n \times [0, T])$ of the stochastic differential equation

$$
\begin{aligned}
dX =& b(X, t)dt + B(X, t)dW \;\; (0 \leq t \leq T) \\
X(0) =& X_0.
\end{aligned}
$$

Theorem A.2 (Rule of de l'Hôpital)**.**
Suppose f and g are differentiable on $(b,c)\setminus\{a\}$, where $a\in\mathbb{R}$, $b\in\mathbb{R}$, $c\in\mathbb{R}$ and $b<a<c$. Suppose either $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ or $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=\infty$. Suppose, in both cases that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists. Then the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ also exists and is given by

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

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