On the Valuation of Fader and Discrete Barrier Options in Heston's Stochastic Volatility Model

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Abstract. We focus on closed-form option pricing in Heston's stochastic volatility model, where closed-form formulas exist only for a few option types. Most of these closed-form solutions are constructed from characteristic functions. We follow this closed-form approach and derive multivariate characteristic functions depending on at least two spot values for different points in time. The derived characteristic functions are used as building blocks to set up (semi-) analytical pricing formulas for exotic options with payoffs depending on finitely many spot values such as fader options and discretely monitored barrier options. We compare our result with different numerical methods and examine accuracy and computational times.

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1 Introduction to the Heston Model

The stochastic volatility model of Heston is characterized by the following system of stochastic differential equations as

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{v_t}dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v \end{aligned} \tag{1}$$

with

$$dW_t^S dW_t^v = \rho dt.$$

The processes $\{S_t\}_{t\geq 0}$ and $\{v_t\}_{t\geq 0}$ denote the spot price and instantaneous variance, respectively. The variance process $\{v_t\}$ is driven by a mean-reverting stochastic square-root process. The two Wiener processes $\{W^S\}$ and $\{W^v\}$ are correlated with correlation rate ρ. In a Foreign Exchange (FX) setting the risk-neutral drift term r of the underlying price process is set to the difference between the domestic and foreign interest rates $r_d - r_f$.

All five parameters of the Heston model, i.e., the long term variance θ, the rate of mean reversion κ, the volatility of variance σ, the correlation ρ and the initial variance v_0 are assumed to be constant and satisfy

$$\theta > 0, \quad \kappa > 0, \quad \sigma > 0, \quad |\rho| < 1, \quad v_0 \geq 0. \tag{2}$$

The term $\sqrt{v_t}$ in the equations (1) ensures the use of non-negative volatility in the spot price process in a continuous theory. It is well-known that the distribution of values of the variance process is given by a non-central chi-squared distribution. This distribution is defined on the non-negative real line and hence, the probability that the variance takes a negative value is equal to zero. So, if the process touches the zero bound, the stochastic part of the volatility process becomes zero and because of the positivity of κ and θ the deterministic part will ensure a non-negative volatility.

Stochastic volatility models are useful because they explain the "volatility smile", the empirical phenomenon that options with different moneyness and expirations have different Black-Scholes implied volatilities. More interestingly, the values of exotic options given by models based on Black-Scholes assumptions can deviate significantly from market prices and option traders are motivated to find models that can take the volatility smile into account. In respect thereof pricing methods for exotic options in stochastic volatility models need to be developed.

1.1 Option Pricing in the Heston Model

In the Black-Scholes model, there is only one source of randomness in the spot price process and contingent claims can be hedged by trading in the money market and the underlying security itself. Whereas in the Heston model case, random changes in volatility also need to be hedged in order to form a self-financing hedge portfolio and therefore to price contingent claims by the no-arbitrage principle. Thus, to achieve this kind of model "completeness" (in the sense that every contingent claim can be replicated by a self-financing trading strategy in the underlying securities) in the Heston model, we assume that additionally to trading in the money market and the underlying security, we can trade in another liquid security, which depends on time, volatility and the underlying spot price process. With these three basic securities, we can set up a self-financing hedge portfolio which replicates a general contingent claim with value function $V(t, v, S)$.

As shown by Hakala and Wystup [13] in a Foreign Exchange setting, the value function V satisfies

$$\begin{aligned} 0 &= V_t + (\kappa\theta - (\kappa+\lambda)v)V_v + (r_d - r_f)SV_S \\ &\quad + \frac{1}{2}\sigma^2 v V_{vv} + \frac{1}{2}vS^2 V_{SS} + \rho\sigma v S V_{vS} - r_d V, \end{aligned}$$

in the region $0 \leq t \leq T$, $0 < S < \infty$ and $0 \leq v < \infty$. The variable λ is used to denote the market price of volatility risk, which is set to zero in this paper without loss of generality. A solution to the above equation can be obtained by specifying appropriate exercise and boundary conditions, which depend on the contract specification.

1.2 Numerical Pricing Methods versus (Semi-) Analytical Pricing Formulas

In stochastic volatility models in general, options can be priced using analytical formulas or numerical methods. Numerical pricing of exotic options in the Heston model can be carried out using conventional numerical methods such as Monte Carlo simulation, finite differences, tree methods or an exact simulation method. Monte Carlo simulation in the Heston model has been explored, for example, by Andersen [2], Higham and Mao [15] and Lord et. al. [23]. An introduction to finite difference methods in the Heston model is given in [20] by Kluge. A method to simulate logarithmic spot values with respect to its exact probability distribution was developed by Broadie and Kaya in [4]. When evaluating exotic options with numerical methods one faces two difficulties. First, depending on which exotic option to price, choosing the adequate numerical method, and second, once the method is selected, how to deal with the challenges of the numerical method itself.

Monte Carlo simulation, for instance, is a robust and strong method which can be used for pricing almost every - especially path-dependent - option. But in the Heston model, two aspects have to be taken into account, if Monte Carlo is the numerical method of choice. One aspect is that the use of Monte Carlo methods in the Heston model depends on the choice of the model parameters κ, θ, σ and v_0. Discretization of the variance process with an Euler scheme, for example, with times u and t, $u < t$, leads to

$$v_t = v_u + \kappa(\theta - v_u)(t-u) + \sigma\sqrt{v_u} z \sqrt{t-u}, \quad z \sim \mathcal{N}(0,1).$$

It follows, that by discretizing this process we modify the probability of obtaining a negative value for the variance. As Lord et al. point out in [23] by using Euler discretization we change it from zero to something normally distributed and therefore positive with probability

$$\mathbb{P}(v_t < 0) = N\left(\frac{-v_u - \kappa(\theta - v_u)(t-u)}{\sigma\sqrt{v_u(t-u)}}\right).$$

Higham and Mao [15] and Lord, Koekkoek and van Dijk [23] deal with this just described problem by setting up various first and second order discretization schemes for the volatility process and by investigating convergence and approximation aspects of the resulting vanilla and barrier option prices. One possible solution to this problem would be to find a discretization scheme which does not change the probability of negative variance values and still maintains the speed of simulating with an Euler scheme.

Hence, although a number of efficient numerical methods to compute option values is available, it is advantageous to have analytical solutions for the value of a financial instrument within a given model - as the solutions obtained will be exact and can be used as a benchmark. Furthermore, the available methods to compute them work independent of the model contrary to numerical simulation methods. For example, the use of Monte Carlo methods in the Heston model for the variance process is critical because of the Lipschitz continuity condition. Whereas, numerical methods to approximate integrals such as in (3) below, just like numerical integration or fast Fourier transforms, can be used in full generality, since they are techniques which are employed and explored in a wide field of applications. Applying these methods, we can benefit from the research advances made in this area and of the important fact that they are not dependent on the choice of the parameter set in the Heston model – Feller's stability condition[1] $2\kappa\theta/\sigma^2 \geq 1$ is no longer a constraint on the model parameters.

Closed-form option valuation in the Heston model has so far been limited to a few option types. Heston provided a closed-form solution for European vanilla options in his original paper [14]. The call value at a time $t < T$ with maturity T and strike price K is given by

$$\text{Call} \quad = \quad e^{-r_f(T-t)} S_t P_S - K e^{-r_d(T-t)} P_N, \tag{3}$$

where for $j = N, S$

$$P_j \quad = \quad \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{\exp(-iu\ln K)\varphi_j(u)}{iu}\right] du. \tag{4}$$

The function $\varphi_j(u) = \exp(B_j(u) + A_j(u)v_t + iu\ln S_t)$ denotes the characteristic function of the random variable $\ln S_T$ at time t under two different measures ($j = N, S$). The functions A and B depend on the time to maturity $T-t$, interest rates r_d, r_f and the set of model parameters κ, ρ, θ, σ.

[1] If $2\kappa\theta/\sigma^2 < 1$, assuming that $v_0 > 0$, the origin is accessible and strongly reflecting. That is why in this situation the probability of hitting zero is quite significant and the process v often has a strong affinity for the area around the origin (see Andersen [2]). Simulating this process at discrete time points therefore leads frequently to the problem of generating negative volatility values.

Some other closed-form solutions for various types of options in the Heston model have been found by a number of researchers:

- Grünbichler and Longstaff [12], 1996: Volatility Option
  The transition density of the volatility process is known to be a non-central chi-squared distribution.

- Dempster and Hong [9], 2000: Correlation Option
  The characteristic function of two spot prices at maturity is derived in a 2-factor model with stochastic volatility.

- Zhu [29], 2000: Exchange, Chooser and Product Option, Barrier Option on Futures for $\rho = 0$
  Formulas for the above mentioned options are derived via the characteristic functions of $\ln S_T$.

- Faulhaber and Lipton [11], 2001: Double Barrier Option for $\rho = 0$ and $r_d = r_f$
  Two methods are presented to derive analytical solutions for this special class of path-dependent options: the method of images and the eigenfunction expansion approach. It was shown that a generalization for Heston's model without the above restrictions ($\rho = 0$ and $r_d = r_f$) fails for both methods.

- Kruse and Nögel [21], 2004: Forward Start Option
  The derivation is based on the fact that at the determination time of the strike, the option price probabilities are not dependent on the actual spot price. Therefore, the formulas are derived by solving expectations via the transition density of v.

- Chiarella and Ziogas [6], 2006: American option
  The pricing problem is formulated as the solution to an inhomogeneous partial differential equation. The corresponding homogeneous problem is solved using Laplace and Fourier transforms and this solution is extended to the solution to the inhomogeneous case with the application of Duhamel's principle. An integral equation is provided for the early exercise region of the option.

Summing up we may say that, so far, closed-form formulas in the Heston model mostly exist for options which are dependent on one spot value at maturity, $\ln S_T$, on values of the volatility at intermediate dates, $v_{t_1}, \ldots, v_{t_n}$, or are only valid in a reduced Heston model framework with uncorrelated Brownian motions, $\rho = 0$. The recent results for the forward start and American option provide formulas for options with a payoff dependent on the path of the spot price and are in the line of this work. We extend the above list of applications of option valuation under the Heston's stochastic volatility dynamics to include weakly path-dependent products.

1.3 Results of this Paper and Outline

With Heston's formula (3) and the formulas in Zhu [29] we can identify a general format of a certain type of closed-form solutions in the Heston model. These solutions are essentially based on probabilities like $\mathbb{P}(S > c)$, where c is a constant and S some random spot value. These probabilities can be expressed in terms of distribution functions $F(c)$, which in turn can be determined by evaluating Fourier integrals with respect to characteristic functions, as in the case of call options in (4). We make use of this observation to establish (semi-)analytical formulas for exotic options with a payoff function, which depends on finitely many spot price values at fixed times $0 < t_1 < \ldots < t_n$ in the following respect

$$\text{Payoff}(S_{t_1}, \ldots, S_{t_n}) \quad = \quad (\pm(S_{t_n} - K))^+ \times f\left(\mathbb{1}_{\{S_{t_i} \lessgtr b_i\}} \mathbb{1}_{\{v_{t_i} \lessgtr b_i\}}\right). \tag{5}$$

The function f defines a combination of indicators $\mathbb{1}_{\{S_{t_i} \leq b_i\}}$, $\mathbb{1}_{\{S_{t_i} \geq b_i\}}$, $\mathbb{1}_{\{v_{t_i} \geq c_i\}}$ or $\mathbb{1}_{\{v_{t_i} \leq c_i\}}$ $(i = 1, \ldots, n)$ with respect to the operations $-$, $\times$ and $+$ and the boundaries b_i and c_i are deterministic. Fader options and discrete barrier options are indicative examples of such combinations. Therefore, we derive multivariate characteristic functions, which allow us to compute values of options of type (5) in closed form.

The remaining part of this paper is organized as follows: In section 2, we derive multivariate characteristic functions dependent on random future values of the logarithmic spot. This result plays a central role throughout this paper, since its existence in closed-form enables us to apply it to the valuation of exotic options, in particular fader options and discrete barrier options. These options are discussed in sections 3 and 4. We consider the general problem of evaluating these claims through a model independent formula (with respect to an equivalent martingale measure) and apply the results which were derived in the previous sections to obtain solutions for the valuation problem in the Heston model. In section 5, we discuss the calculation of the probabilities contained in the established analytical formulas and present numerical examples.

2 Characteristic Functions

In this section, we derive n-variate characteristic functions of the log-spot prices $\ln S_{t_1}, \ldots, \ln S_{t_n}$ at times $0 < t_1 < \ldots < t_n = T$ in the Heston model under two different probability measures. This result is used to establish closed-form valuation formulas for various exotic options in sections 3 and 4.

2.1 Derivation of the n-variate Characteristic Function

Let $X = (X_1, \ldots, X_n)'$ be a random vector and $u = (u_1, \ldots, u_n)$ be a vector of real numbers. The joint characteristic function of n random variables $X_1, \ldots, X_n$ is defined by

$$\varphi_X(u) \quad = \quad \mathbb{E}\left[e^{iuX}\right] = \int_{\mathbb{R}^n} \exp\left(iu_1 x_1 + \ldots + iu_n x_n\right) d\mathbb{P}^X,$$

where $\mathbb{P}^X$ is the probability measure function of X. The function $\varphi_X(u)$ is a complex-valued continuous function of the n real variables $u_1, \ldots, u_n$. We derive the characteristic function under the risk-neutral measure $\mathbb{Q}_N$ and the spot measure $\mathbb{Q}_S$ with the spot price as numeraire.

Theorem 1 *In the Heston model as defined in (1) the joint characteristic function of the logarithm of spot values $X = (x_{t_1}, \ldots, x_{t_n})$ at times $0 = t_0 < t_1 < \ldots < t_n = T$ under the risk-neutral measure $\mathbb{Q}_N$ is given by*

$$\varphi_X^N(u_1, \ldots, u_n) = \exp\left(\sum_{k=1}^{n} iu_k h(t_k) - \sum_{k=1}^{n} q(u_k) j(t_k) + \sum_{k=1}^{n} B_k + A_n v_0\right) \quad (6)$$

where we set

$$B_k = B\left(t_{n-k+1} - t_{n-k}, q(u_{n-k+1}) + A_{k-1}, p\left(\sum_{j=n-k+1}^{n} u_j\right)\right) \quad (7)$$

$$A_k = A\left(t_{n-k+1} - t_{n-k}, q(u_{n-k+1}) + A_{k-1}, p\left(\sum_{j=n-k+1}^{n} u_j\right)\right) \quad (8)$$

starting with $A_0 = 0$ and[2]

$$h(t) = x_0 + (r_d - r_f)t \quad (9)$$

$$j(t) = v_0 + \kappa\theta t. \quad (10)$$

The functions A and B above are defined as

$$A(\tau) = A(\tau, a, b) = \frac{da\left(1 + e^{-d\tau}\right) - \left(1 - e^{-d\tau}\right)(2b + \kappa a)}{\gamma} \quad (11)$$

$$B(\tau) = B(\tau, a, b) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)\tau + \frac{2\kappa\theta}{\sigma^2} \ln \frac{2d}{\gamma} \quad (12)$$

with

$$d = \sqrt{\kappa^2 + 2\sigma^2 b}$$

$$\gamma = d\left(1 + e^{-d\tau}\right) + (\kappa - \sigma^2 a)\left(1 - e^{-d\tau}\right)$$

and the functions p and q as

$$p(u) = \left(\frac{1}{2} - \kappa\frac{\rho}{\sigma} - \frac{1}{2} iu(1 - \rho^2)\right) iu \quad (13)$$

$$q(u) = iu\frac{\rho}{\sigma}. \quad (14)$$

The n-variate characteristic function under the spot measure $\mathbb{Q}_S$ is given by

$$\varphi_X^S(u_1, \ldots u_n) = \exp\left(\sum_{k=1}^{n} iu_k h(t_k) - \sum_{k=1}^{n} q(u_k) j(t_k) - \frac{\rho}{\sigma} j(t_n) + \sum_{k=1}^{n} B_k + A_n v_0\right) \quad (15)$$

[2] The proof of this theorem can be maintained also with $\lambda \neq 0$.

with a different definition of the functions A and B than for φ^N, namely

$$\begin{aligned} B_k &= B\left(t_{n-k+1} - t_{n-k}, q(u_{n-k+1}) + A_{k-1}, p\left(\sum_{j=n-k+1}^{n} u_j - i\right)\right) \\ A_k &= A\left(t_{n-k+1} - t_{n-k}, q(u_{n-k+1}) + A_{k-1}, p\left(\sum_{j=n-k+1}^{n} u_j - i\right)\right) \end{aligned}$$

starting with $A_0 = \frac{\rho}{\sigma}$.

Remark 1 *Note that the exponents of the exponential function in φ^N and φ^S are linearly dependent on the state variables at time 0, v_0 and x_0. The functions A and B are defined recursively. Both functions call as an argument the value of A in the previous step. For $n = 1$ the result in theorem 1 reduces to the univariate characteristic function which is used in the closed-form formula for vanilla options by Heston.*

Proof. For $n = 1$ the characteristic functions are known. We use induction, beginning with the characteristic function of two random logarithmic spot values at two different points in time t_1 and t_2 with $0 < t_1 < t_2$.

Let x_t denote the logarithmic spot value $\ln S_t$ at an arbitrary time $0 < t \leq t_2$. Then the logarithmic spot price at time t_1, given the values x_0 and v_0, can be written as

$$\begin{aligned} x_{t_1} &= x_0 + (r_d - r_f)t_1 - \frac{1}{2}\int_0^{t_1} v_t dt + \int_0^{t_1} \sqrt{v_t} dW_t^S \\ &= x_0 + (r_d - r_f)t_1 - \frac{1}{2}\int_0^{t_1} v_t dt + \rho\int_0^{t_1} \sqrt{v_t} dW_t^v + \rho_2 \int_0^{t_1} \sqrt{v_t} dW_t, \end{aligned} \tag{16}$$

where $\rho_2 = \sqrt{1-\rho^2}$ and $dW_t^S = \rho dW_t^v + \rho_2 dW_t$ is the Cholesky decomposition of the Brownian motion W^S into the sum of W^v and another independent Brownian motion W.

The variance at time t_1 is given by the integral equation

$$v_{t_1} - v_0 = \kappa\theta t_1 - \kappa\int_0^{t_1} v_t dt + \sigma\int_0^{t_1} \sqrt{v_t} dW_t^v. \tag{17}$$

The goal is to derive the characteristic function for two different measures, the risk-neutral $\mathbb{Q}_N$ and the spot measure $\mathbb{Q}_S$ with S as its numeraire. We denote the Radon-Nikodym derivatives corresponding to the measures $\mathbb{Q}_N$ and $\mathbb{Q}_S$ by

$$g_N(t_2) = 1, \qquad g_S(t_2) = \exp\left(-(r_d - r_f)t_2 + x_{t_2} - x_0\right), \tag{18}$$

and obtain the bivariate characteristic function φ_X^j, $j = N, S$, for $X = (x_{t_1}, x_{t_2})$ under the measures $\mathbb{Q}_N$ and $\mathbb{Q}_S$ by

$$\varphi_X^j(u_1, u_2) = \mathbb{E}^{\mathbb{Q}_j}\left[\exp(iu_1 x_{t_1} + iu_2 x_{t_2})\right]. \tag{19}$$

The derivation of φ^N and φ^S is similar, since

$$\varphi_X^S(u_1,u_2) = \exp(-(r_d-r_f)t_2-x_0)\mathbb{E}^{\mathbb{Q}_N}\left[\exp(iu_1x_{t_1}+i(u_2-i)x_{t_2})\right]. \tag{20}$$

We proceed with the derivation of φ^N.

Invoking equation (17), we can replace the term $\int_0^{t_1}\sqrt{v_t}dW_t^v$ in equation (16) by

$$\frac{1}{\sigma}\left[v_{t_1}-v_0-\kappa\theta t_1+\kappa\int_0^{t_1}v_tdt\right].$$

Inserting the model definitions for x_{t_1} and x_{t_2} into (19) we derive

$$\begin{aligned}
&\varphi_X^N(u_1,u_2)\\
=\ &\exp(iu_1h(t_1)+iu_2h(t_2))\\
&\mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{i(u_1+u_2)\left(-\frac{1}{2}\int_0^{t_1}v_tdt+\frac{\rho}{\sigma}\left[v_{t_1}-j(t_1)+\kappa\int_0^{t_1}v_tdt\right]+\rho_2\int_0^{t_1}\sqrt{v_t}dW_t\right)\right.\right.\\
&\left.\left.+iu_2\left(-\frac{1}{2}\int_{t_1}^{t_2}v_tdt+\frac{\rho}{\sigma}\left[v_{t_2}-v_{t_1}-\kappa\theta(t_2-t_1)+\kappa\int_{t_1}^{t_2}v_tdt\right]+\ \rho_2\int_{t_1}^{t_2}\sqrt{v_t}dW_t\right)\right\}\right],
\end{aligned}$$

with h and j defined as in (9) and (10), respectively. Let $\sigma(W_s^v : 0 \le s \le t_2)$ represent the filtration generated by $\{W_s^v\}_{t\le s\le t_2}$. In the following step, we take the conditional expectation value with respect to $\sigma(W_s^v : 0 \le s \le t_2)$. Since all terms in the expectations are W^v-measurable except the ones containing $iu_2\rho_2\int_{t_1}^{t_2}\sqrt{v_t}dW_t$ and $i(u_1+u_2)\rho_2\int_0^{t_1}\sqrt{v_t}dW_t$ we obtain

$$\begin{aligned}
&\varphi_X^N(u_1,u_2)\\
=\ &\exp(iu_1h(t_1)+iu_2h(t_2))\\
&\mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{i(u_1+u_2)\left(-\frac{1}{2}\int_0^{t_1}v_tdt+\frac{\rho}{\sigma}\left[v_{t_1}-j(t_1)+\kappa\int_0^{t_1}v_tdt\right]\right)\right.\right.\\
&\left.+iu_2\left(-\frac{1}{2}\int_{t_1}^{t_2}v_tdt+\frac{\rho}{\sigma}\left[v_{t_2}-v_{t_1}-\kappa\theta(t_2-t_1)+\kappa\int_{t_1}^{t_2}v_tdt\right]\right)\right\}\\
&\left.\mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{i\rho_2(u_1+u_2)\int_0^{t_1}\sqrt{v_t}dW_t+iu_2\rho_2\int_{t_1}^{t_2}\sqrt{v_t}dW_t\right\}\middle|\sigma(W_s^v:0\le s\le t_2)\right]\right]
\end{aligned}$$

Given $\{W^v\}$, the path of v is known from time $t=0$ until t_2, and is therefore deterministic. It follows that the integrals $\int_0^{t_1}\sqrt{v_t}dW_t$ and $\int_{t_1}^{t_2}\sqrt{v_t}dW_t$ are normally distributed with zero mean. Since W^v and W are independent, the two integrals are also uncorrelated and therefore the random variables

$$\exp\left(i(u_1+u_2)\rho_2\int_0^{t_1}\sqrt{v_t}dW_t\right) \quad \text{and} \quad \exp\left(iu_2\rho_2\int_{t_1}^{t_2}\sqrt{v_t}dW_t\right)$$

are independent. Hence, the above expectation is equal to the product of two single expectations of the two terms. The variances are calculated via the Itô isometry, and are equal to $\int_0^{t_1}v_tdt$

and $\int_{t_1}^{t_2} v_t dt$, respectively. Using the characteristic function for a normally distributed variable X, $\mathbb{E}[e^{iaX}] = e^{ia\mathbb{E}X - \frac{1}{2}a^2 \text{Var}X}$, the above yields

$$\begin{aligned}\varphi_X^N(u_1,u_2) &= \exp(iu_1h(t_1) + iu_2h(t_2) - i(u_1+u_2)\frac{\rho}{\sigma}j(t_1) - iu_2\frac{\rho}{\sigma}\kappa\theta(t_2-t_1)) \\ &\quad \mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{iu_1\frac{\rho}{\sigma}v_{t_1} + iu_2\frac{\rho}{\sigma}v_{t_2} + \left(-\frac{1}{2} + \kappa\frac{\rho}{\sigma} + \frac{1}{2}i(u_1+u_2)\rho_2^2\right)\right.\right. \\ &\quad \left.\left. i(u_1+u_2)\int_0^{t_1} v_t dt + \left(-\frac{1}{2} + \kappa\frac{\rho}{\sigma} + \frac{1}{2}iu_2\rho_2^2\right) iu_2 \int_{t_1}^{t_2} v_t dt\right\}\right].\end{aligned}$$

Using the functions p and q defined in (13) and (14), the characteristic function takes the form

$$\begin{aligned}\varphi_X^N(u_1,u_2) &= \exp(iu_1h(t_1) + iu_2h(t_2) - q(u_1)j(t_1) - q(u_2)j(t_2)) \\ &\quad \mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{q(u_1)v_{t_1} + q(u_2)v_{t_2} + p(u_2)\int_{t_1}^{t_2} v_t dt + p(u_1+u_2)\int_0^{t_1} v_t dt\right\}\right].\end{aligned}$$

Now we see that the characteristic function consists only of two types of random variables: the values of the variance at both times t_1 and t_2, and the time-integrals with respect to the paths of the variance process between 0 and t_1 and between t_1 and t_2. Therefore, using the tower property and taking out the terms which are known with respect to the information up to time t_1 results in

$$\begin{aligned}\varphi_X^N(u_1,u_2) &= \exp(iu_1h(t_1) + iu_2h(t_2) - q(u_1)j(t_1) - q(u_2)j(t_2)) \\ &\quad \mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{q(u_1)v_{t_1} + p(u_1+u_2)\int_0^{t_1} v_t dt\right\}\right. \\ &\quad \left.\mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{q(u_2)v_{t_2} + p(u_2)\int_{t_1}^{t_2} v_t dt\right\}\middle|\mathcal{F}_{t_1}\right]\right].\end{aligned}$$

We notice that the calculation of $\varphi_X^N(u_1,u_2)$ is now reduced to that of the above nested expectations. The inner expectation

$$\mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{q(u_2)v_{t_2} + p(u_2)\int_{t_1}^{t_2} v_t dt\right\}\middle|\mathcal{F}_{t_1}\right]$$

is solvable by application of the Feynman-Kac formula.

If we define the function $y(t,v_t)$ for a fixed time $0 < t < t_2$ by

$$y(t,v_t) = \mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{q(u_2)v_{t_2} + p(u_2)\int_t^{t_2} v_s ds\right\}\middle|\mathcal{F}_t\right]$$

the Feynman-Kac formula tells us that y must satisfy the partial differential equation

$$-\frac{\partial y}{\partial t} = p(u_2)vy + \kappa(\theta - v)\frac{\partial y}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 y}{\partial v^2}$$

with boundary condition

$$y(t_1, v_{t_1}) = \exp(q(u_2)v_{t_1}).$$

This partial differential equation is solvable if we assume that y is log-linear[3] and given by $y(t, v_t) = \exp\left[A(t_2 - t)v_t + B(t_2 - t)\right]$. Then the functions A and B must be of the form (11) and (12), respectively.

Inserting this solution into the outer expectation above, the characteristic function has the following structure

$$\begin{aligned}\varphi_X^N(u_1, u_2) &= \exp(iu_1h(t_1) + iu_2h(t_2) - q(u_1)j(t_1) - q(u_2)j(t_2)) \\ &\quad \exp(B(t_2 - t_1, q(u_2), p(u_2))) \\ &\quad \mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{(q(u_1) + A(t_2 - t_1, q(u_2), p(u_2)))\, v_{t_1} + p(u_1 + u_2)\int_0^{t_1} v_t dt\right\}\right].\end{aligned}$$

It remains to solve the outer expectation in $\varphi_X^N(u_1, u_2)$

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}_N}\left[\exp\left\{(q(u_1) + A(t_2 - t_1, q(u_2), p(u_2)))v_{t_1} + p(u_1 + u_2)\int_0^{t_1} v_t dt\right\}\right] \\ =\ &\exp\left[A(t_1, A_1, p(u_1 + u_2))v_0 + B(t_1, A_1, p(u_1 + u_2))\right] \\ =\ &\exp\left[A_2 v_0 + B_2\right],\end{aligned}$$

where A_1, A_2 and B_2 are defined in equations (8) and (7).

Therefore, the joint characteristic function of $\ln S_{t_1}$ and $\ln S_{t_2}$ with respect to the probability measure $\mathbb{Q}_N$ is given by

$$\varphi_X^N(u_1, u_2) = \exp\Big(i(u_1h(t_1) + u_2h(t_2)) - q(u_1)j(t_1) - q(u_2)j(t_2) + B_1 + B_2 + A_2v_0\Big),$$

with the functions A_2, B_1 and B_2 defined as in (8) and (7).

By repeated application of the same principles as in the derivation above we can show that by induction that the n-variate characteristic functions under the measures $\mathbb{Q}_N$ and $\mathbb{Q}_S$ of the log-spot vector $X = (x_{t_1}, \ldots, x_{t_n})$ at times $0 < t_1 < \ldots < t_n = T$ for an arbitrary n are given by (6) and (15). □

Remark 2 *The same idea can be used to derive multivariate characteristic functions dependent on n log-spot values and m volatility values.*

Remark 3 *The derivation of the n-variate characteristic functions can be adapted and transfered to a more general class of stochastic volatility models. Further examples of these kind of models are the model of Schöbel & Zhu in [29], the Bates (SVJ) and SVCJ model in Duffie et. al. [10] and multidimensional Heston models like the three-factor model mentioned in Dempster & Hong [9] or the model developed by Grasselli et. al. [8].*

[3] We set up the derivatives of y w.r.t. τ, v and v^2 and then solve the resulting Riccatti-type ordinary differential equations.

2.2 Applications of Characteristic Functions in Option Pricing

The result following below in this section makes the important theoretical connection between characteristic functions and distribution functions in analytical form. This enables us to derive closed-form formulas in (in)complete models. We suppose that the characteristic function φ is known, as in (6) and (15), and we wish to compute the distribution function F directly from it.

Theorem 2 (Shephard's Theorem) *Let F denote the distribution function of interest. Suppose its corresponding density, f, is Lebesgue-integrable, $f \in L^n$, and its characteristic function $\varphi(u) \in L^n$. Then under the assumption of the existence of a mean for the random variable of interest, the following equality holds for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$:*

$$\begin{aligned} t(x) &= 2^n F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) - 2^{n-1}[F_{X_2,\ldots,X_n}(x_2,\ldots,x_n) + \cdots + F_{X_1,\ldots,X_{n-1}}(x_1,\ldots,x_{n-1})] \\ &\quad +2^{n-2}[F_{X_3,\ldots,X_n}(x_3,\ldots,x_n) + \cdots + F_{X_1,\ldots,X_{n-2}}(x_1,\ldots,x_{n-2})] + \cdots + (-1)^n, \end{aligned}$$

where we define

$$t(x) = \frac{(-2)^n}{(2\pi)^n} \int_0^\infty \cdots \int_0^\infty \Delta_{u_1}\left[\cdots \Delta_{u_n}\left[\frac{\varphi(u)e^{-ix^\perp u}}{iu_1 \cdots iu_n}\right]\right] du, \tag{21}$$

with $u = (u_1, \ldots, u_n)^\perp$ and $\Delta_a[\eta(a)] = \eta(a) + \eta(-a)$.

Proof: The proof is given in Shephard [27].

Remark 4 *The result of theorem 2 above can be specified for the cases of n being odd or even*

$$\Delta_{u_1}\left[\cdots \Delta_{u_n}\left[\frac{\varphi(u)e^{-ix^\perp u}}{iu_1 \cdots iu_n}\right]\right] = \begin{cases} 2i^{n-1}\Delta_{u_2}\left[\cdots \Delta_{u_n}\Im\left[\frac{\varphi(u)e^{-ix^\perp u}}{u_1\cdots u_n}\right]\right], & \text{if } n \text{ is odd} \\ 2i^n\Delta_{u_2}\left[\cdots \Delta_{u_n}\Re\left[\frac{\varphi(u)e^{-ix^\perp u}}{u_1\cdots u_n}\right]\right], & \text{if } n \text{ is even.} \end{cases}$$

For an implementation it might be better to express it with respect to the real part

$$\Delta_{u_1}\left[\cdots \Delta_{u_n}\left[\frac{\varphi(u)e^{-ix^\perp u}}{iu_1 \cdots iu_n}\right]\right] = 2\Delta_{u_2}\cdots\Delta_{u_n}\Re\left[\frac{\varphi(u)e^{-ix^\perp u}}{iu_1 \cdots iu_n}\right].$$

These results indicate how to calculate an n-dimensional distribution function if the n-variate characteristic function is given: compute recursively all values for the marginal distribution functions and then the integral term in (21). In particular, by definition of the distribution function we are able to compute values for probabilities $\mathbb{P}(S_{t_1} \leq c_1, \ldots, S_{t_n} \leq c_n)$ with constant boundaries $c_i, i = 1, \ldots, n$. All other probabilities such as $\mathbb{P}(S_{t_1} \geq c_1, \ldots, S_{t_n} \geq c_n)$ can also be calculated if we express the probability in terms of distribution functions F, for example,

$$\mathbb{P}(S_{t_1} \geq c_1, \ldots, S_{t_n} \geq c_n) = 1 - \sum_{i=1}^n F_{S_{t_i}}(c_i) + \sum_{i,j} F_{S_{t_i},S_{t_j}}(c_i, c_j) \pm \ldots \pm F_{S_{t_1},\ldots,S_{t_n}}(c_1, \ldots, c_n).$$

Determining probabilities of such events establishes the core problem for the valuation of weakly path-dependent options. For instance, the computation of probabilities $\mathbb{P}(S_{t_1} \le c_1, \ldots, S_{t_n} \le c_n)$ is part of the computation of values of discretely monitored up-and-out options, where the distribution of the random variables $S_{t_1}, \ldots, S_{t_n}$ is defined by the model at hand and determined by their joint characteristic function. Similarly, probabilities of the form $\mathbb{P}(S_{t_1} \ge c_1, \ldots, S_{t_n} \ge c_n)$ need to be calculated for the valuation of other options, such as discrete down-and-out options.

The above remarks show that the application of Shephard's theorem might be useful only for lower dimensional problems. As the formula for an n-dimensional distribution function F contains all marginal distribution functions, it can be computationally time consuming to evaluate them with multidimensional numerical integration methods. Therefore, this method might be only suitable for the valuation of options which are dependent on a small number of random spot values. In the next section we will apply it for the case of fader options, a case where the payoff of the option depends on two random variables $X_1 = \ln S_t$ and $X_2 = \ln S_T$ $(n = 2)$. Then the above statement yields the relationship

$$\begin{aligned} \frac{2^2}{(2\pi)^2} \iint_0^\infty \Delta_{u_1} \left[\Delta_{u_2} \left[\frac{\varphi(u) e^{-ix^\perp u}}{iu_1 iu_2} \right] \right] dt_1 dt_2 &= \frac{-2^3}{(2\pi)^2} \iint_0^\infty \Delta_{u_2} \Re \left[\frac{\varphi(u) e^{-ix^\perp u}}{u_1 u_2} \right] du_1 du_2 \\ &= 4F_{X_1,X_2}(x_1,x_2) - 2[F_{X_1}(x_1) + F_{X_2}(x_2)] + 1. \end{aligned}$$

Therefore, the distribution function F of X_1 and X_2 at (x_1, x_2) is given by

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= \frac{1}{4} - \frac{1}{2\pi} \int_0^\infty \Re \left[\frac{\varphi(u_1,0) e^{-iu_1 x_1} + \varphi(0,u_2) e^{-iu_1 x_2}}{iu_1} \right] du_1 \\ &\quad - \frac{1}{2\pi^2} \iint_{\mathbb{R}_+^2} \Re \left[\frac{\varphi(u_1,u_2) e^{-iu_1 x_1 - iu_2 x_2} - \varphi(u_1,-u_2) e^{-iu_1 x_1 + iu_2 x_2}}{u_1 u_2} \right] du_1 du_2. \end{aligned} \tag{22}$$

The closed-form pricing formula for fader options is composed out of these and similar probabilities.

Another possible application of the results of theorem 1 is using (fractional) fast Fourier transforms, as we will discuss in section 5.

3 Fader Options

A fader option is a plain vanilla option whose notional is determined by a fade-in (or fade-out) factor λ. This factor λ increases (decreases) for every time t_i where the spot fixing stays inside a given range $[L, H]$. If the spot never leaves the range, in the case of a fade-in option the payoff is a plain vanilla payoff with 100% of the notional accumulated. More formally, the payoff of a fade-in call at maturity T is given by

$$\lambda \left(S_T - K \right)^+, \quad \text{with } \lambda = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{ S_{t_i} \in [L,H] \right\}},$$

where $0 < t_1 < \cdots < t_N = T$ is a set of fade-in dates within $[0, T]$. We take spot values at time t as a usual approximation of the fixing or closing price. The impact of this approximation is illustrated in Becker and Wystup [3]. For fade-out options λ is replaced with $1 - \lambda$.

The advantage of a fade-in option is that it is cheaper than the corresponding plain vanilla product. However, this kind of product needs incorporation of a market view on the whole spot price path at times t_i. This market view may either be that λ is expected to be close to or smaller than 1. In the first case the factor will not affect the payoff, but will effect the price of the product.

The valuation of fader options in the Black-Scholes model is explained in Overhaus et al. [25] and Hakala and Wystup [13]. Various applications in structuring and variations and a trader's approach how to price and hedge a fader option is covered in Wystup [28].

Under the risk-neutral measure $\mathbb{Q}_N$ a fader option with strike price K and fixing times $t_1, \ldots, t_n = T$ can be valued at time 0 in the context of equivalent martingale measures as

$$
\begin{aligned}
V_{Fader}(K, L, H) &= e^{-r_d T} \mathbb{E}^{\mathbb{Q}_N} \left[(S_T - K)^+ \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{S_{t_i} \in [L,H]\}} \right] \\
&= \frac{1}{N} \sum_{i=1}^{N} \underbrace{e^{-r_d T} \mathbb{E}^{\mathbb{Q}_N} \left[(S_T - K)^+ \mathbb{1}_{\{S_{t_i} \in [L,H]\}} \right]}_{= V_F(t_i)} \qquad (23) \\
&= \frac{1}{N} \sum_{i=1}^{N} V_F(t_i).
\end{aligned}
$$

Therefore the valuation of a fader option reduces to the determination of the discounted expectations in equation (23), denoted by $V_F(t)$, for $t \in \{t_1, \ldots, t_N\}$. In the following section we first set up a pricing formula and then derive a closed-form solution for V_F in the Heston model for an arbitrary fixing time $t \leq T$.

3.1 Valuation of Fader Options in the Heston Model

With a change of notation to log-spot values $x_t = \ln S_t$, the value $V_F(t)$, defined in equation (23), is given by

$$
\begin{aligned}
V_F(t) &= e^{-r_d T} \mathbb{E}^{\mathbb{Q}_N} \left[(S_T - K)^+ \mathbb{1}_{\{S_t \in [L,H]\}} \right] \\
&= e^{-r_d T} \mathbb{E}^{\mathbb{Q}_N} \left[(e^{x_T} - K) \mathbb{1}_{\{l \leq x_t \leq h, k \leq x_T\}} \right],
\end{aligned}
$$

and can be extended into the four expectations of indicator functions, so that

$$
\begin{aligned}
V_F(t) = \; & e^{-r_d T} \Big[\mathbb{E}^{\mathbb{Q}_N} \left[e^{x_T} \mathbb{1}_{\{x_t \leq h, x_T \geq k\}} \right] - \mathbb{E}^{\mathbb{Q}_N} \left[e^{x_T} \mathbb{1}_{\{x_t \leq l, x_T \geq k\}} \right] \Big] \\
& - e^{-r_d T} K \Big[\mathbb{E}^{\mathbb{Q}_N} \left[\mathbb{1}_{\{x_t \leq h, x_T \geq k\}} \right] - \mathbb{E}^{\mathbb{Q}_N} \left[\mathbb{1}_{\{x_t \leq l, x_T \geq k\}} \right] \Big], \qquad (24)
\end{aligned}
$$

where $k = \ln K$, $l = \ln L$ and $h = \ln H$. For the first two terms in (24), choose the spot price as numeraire and switch from probability measure $\mathbb{Q}_N$ to $\mathbb{Q}_S$. According to Girsanov's theorem, the relationship between to measures $\mathbb{Q}_N$ and $\mathbb{Q}_S$ is given by the Radon-Nikodym derivative g_S as defined in equation (18).

Under this new measure, the option value representation can be restated as

$$V_F(t) = S_0 \mathbb{E}^{\mathbb{Q}_S}\left[\mathbb{1}_{\{x_T \geq k, x_t \in [l,h]\}}\right] - e^{-r_d T} K \mathbb{E}^{\mathbb{Q}_N}\left[\mathbb{1}_{\{x_T \geq k, x_t \in [l,h]\}}\right].$$

The value of a fadlet $V_F(t)$ in (23), for some $t \in \{t_1, \dots, t_n\}$, can also be expressed in terms of four probabilities, so that

$$\begin{aligned} V_F(t) &= e^{-r_f T} S_0 \Big[\mathbb{Q}_S\left(x_T \geq k, x_t \leq h\right) - \mathbb{Q}_S\left(x_T \geq k, x_t \leq l\right)\Big] \\ &\quad - e^{-r_d T} K \Big[\mathbb{Q}_N\left(x_T \geq k, x_t \leq h\right) - \mathbb{Q}_N\left(x_T \geq k, x_t \leq l\right)\Big]. \end{aligned} \tag{25}$$

In section 2, theorem 2 states the representation of an n-distribution function F in terms of its marginal distribution functions. In case of the fader option we see from equation (25) that we need to be able to compute probabilities of the form $\mathbb{P}(x_t \leq c_1, x_T \geq c_2)$, for some constants c_1 and c_2, with respect to the measures $\mathbb{Q}_N$ and $\mathbb{Q}_S$ in order to price fader options in the Heston model. We apply Shephard's theorem 2 for $n = 2$ (see also equation (22)) to obtain an expression for the 2-dimensional distribution function $F(c_1, c_2)$ with respect to $\mathbb{Q}_j$, $j = N, S$, which is equal to

$$\begin{aligned} F_j(h,k) &= \frac{1}{4} - \frac{1}{2\pi}\int_0^\infty \Re\left[\frac{\varphi_j(0,u_2)e^{-iu_2 k}}{iu_2}\right] du_2 - \frac{1}{2\pi}\int_0^\infty \Re\left[\frac{\varphi_j(u_1,0)e^{-iu_1 h}}{iu_1}\right] du_1 \\ &\quad - \frac{1}{2\pi^2}\iint_{\mathbb{R}_+^2} \Re\left[\frac{\varphi_j(u_1,u_2)e^{-iu_1 h - iu_2 k} - \varphi_j(u_1,-u_2)e^{-iu_1 h + iu_2 k}}{u_1 u_2}\right] du_1 du_2. \end{aligned}$$

Since the joint distribution function $F(c_1, c_2)$ of a random vector $X = (X_1, X_2)$ is defined by the probability $\mathbb{P}(X_1 \leq c_1, X_2 \leq c_2)$, we can express the probability $\mathbb{P}(X_1 \leq c_1, X_2 \geq c_2)$ in terms of distribution functions as

$$\begin{aligned} F^*(c_1,c_2) &= \mathbb{P}(X_1 \leq c_1, X_2 \geq c_2) \\ &= \mathbb{P}(X_1 \leq c_1) - \mathbb{P}(X_1 \leq c_1, X_2 \leq c_2) = F(c_1) - F(c_1,c_2). \end{aligned} \tag{26}$$

From (26), we obtain the desired probabilities by using

$$\begin{aligned} F_j^*(h,k) &= \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty \Re\left[\frac{\varphi_j(0,u_2)e^{-iu_2 k}}{iu_2}\right] du_2 - \frac{1}{2\pi}\int_0^\infty \Re\left[\frac{\varphi_j(u_1,0)e^{-iu_1 h}}{iu_1}\right] du_1 \\ &\quad + \frac{1}{2\pi^2}\iint_{\mathbb{R}_+^2} \Re\left[\frac{\varphi_j(u_1,u_2)e^{-iu_1 h - iu_2 k} - \varphi_j(u_1,-u_2)e^{-iu_1 h + iu_2 k}}{u_1 u_2}\right] du_1 du_2, \end{aligned}$$

which is obtained by an application of Shephard's theorem. Finally, the value of a fader call option at time $t = 0$ in the Heston model is given by

$$\begin{aligned} V_{Fader}(K, L, H) &= \frac{1}{N} \sum_{i=1}^{N} V_F(t_i) \quad \text{with} \\ V_F(t_i) &= S_0 e^{-r_f T} \left[F_2^*(h,k) - F_2^*(l,k) \right] - K e^{-r_d T} \left[F_1^*(h,k) - F_1^*(l,k) \right]. \end{aligned} \tag{27}$$

The corresponding characteristic functions are defined in (6) and (15) by setting n equal to 2. The value of a fader put option can be derived in an equivalent manner.

Note that, the above equation (27) is model independent (within the context of complete models). The calculation of a fader option value within a specific model can be accomplished by calculating the appropriate characteristic functions for $\ln S_t$. For example, to price a fader option in the Black-Scholes model choose the bivariate characteristic function for normally distributed random variables. To price it in the Heston model express F_1^* and F_2^* with respect to the characteristic functions (6) and (15).

Remark 5 *Equation (27) specifies the value of a fader call at time 0 with respect to some underlying distribution of log-spot values and constant market data r_d, r_f, constant contract data K, L, H and constant model parameters. This formula can be extended to a valuation formula for fader options where this data is time-dependent, for example, as step functions taking constant values between fixing times.*

4 Discretely Monitored Barrier Options

One further application of the n-variate characteristic functions is the valuation of discretely monitored barrier options in the Heston model. Barrier options, where the barriers are monitored only at finitely many fixed time points are called discretely monitored barrier options in contrast to continuously monitored barrier options, where the barrier is valid at all times between trade time and maturity. In case of a discretely monitored barrier option with strike K, constant barrier H and maturity T, the payoffs are given by

$$\begin{aligned} &(\phi(S_T - K))^+ \, \mathbb{1}_{\{\max_{i\in\{1,\ldots,n\}} S_{t_i} < H\}} \quad , \quad (\phi(S_T - K))^+ \, \mathbb{1}_{\{\max_{i\in\{1,\ldots,n\}} S_{t_i} > H\}}, \\ &(\phi(S_T - K))^+ \, \mathbb{1}_{\{\min_{i\in\{1,\ldots,n\}} S_{t_i} < H\}} \quad , \quad (\phi(S_T - K))^+ \, \mathbb{1}_{\{\min_{i\in\{1,\ldots,n\}} S_{t_i} > H\}}, \end{aligned}$$

where $\phi = \pm 1$ is a put/call-indicator taking the value $+1$ in case of a call and -1 in case of a put, $0 < t_1 < \ldots < t_n = T$ is a finite set of the barrier monitoring times for the underlying in the time interval $[0, T]$ and T the maturity of the option. The four payoffs above define the payoffs for so called up-and-out, up-and-in, down-and-in and down-and-out options. For calls we abbreviate these payoff functions by UOC, DOC, DIC and UIC, respectively. These notations will also be used to denote the value of the option.

Before going into detail, let us point out the following relations between the payoffs of barrier options and vanilla options. The in-out parity for barrier options, namely, knock-in + knock-out = vanilla, allows us to consider only the family of knock-out options for the derivation of closed-form formulas, since a closed-form formula for vanilla options in the Heston model already exists. Additionally, since the well known symmetry relation between call and put options in the Black-Scholes model can be derived in similar form in the Heston model for discrete barrier options in an FX context, it is enough to treat only calls. Hence, we give details only for knock-out call options. In order to be able to price all types of barrier options, i.e., knock-in calls and puts and knock-out calls and puts, altogether, we need to examine three types of payoff functions; these are

- Down-and-out: For $H < S_0$

$$\begin{cases} (S_T - K)\mathbb{1}_{\{H \le S_{t_1}, \ldots, H \le S_{t_n}\}} & \text{for } K < H \\ (S_T - K)\mathbb{1}_{\{H \le S_{t_1}, \ldots, H \le S_{t_{n-1}}, K \le S_{t_n}\}} & \text{for } H < K \end{cases}$$

- Up-and-out: For $S_0 < H$ and $K < H$

$$(S_T - K)\mathbb{1}_{\{H \ge S_{t_1}, \ldots, H \ge S_{t_{n-1}}, K \le S_{t_n} \le H\}}.$$

Remark 6 *More generally, for each fixed barrier monitoring time t_i there can be a different barrier level H_i. The payoff of an up-and-out call option, for example, then changes to*

$$(S_T - K)^+ \mathbb{1}_{\left\{S_{t_1} < H_1, S_{t_2} < H_2, \ldots, S_{t_n} < H_n\right\}}.$$

Here we choose all barriers to be equal for simplicity and an easier implementation, but of course all the arguments hold also for varying barrier levels H_i.

4.1 Valuation of Discrete Barrier Options in the Heston Model

We can rewrite the value of an up-and-out barrier call option

$$V_{UOC} = e^{-r_d T}\mathbb{E}^{\mathbb{Q}_N}\left[\left(e^{x_T} - K\right)\mathbb{1}_{\{x_T > k\}}\mathbb{1}_{\left\{x_{t_1} < h, \ldots, x_{t_n} < h\right\}}\right] \tag{28}$$

as

$$\begin{aligned} V_{UOC} &= e^{-r_f T} S_0 \mathbb{E}^{\mathbb{Q}_S}\left[\mathbb{1}_{\{x_T > k, x_{t_1} < h, \ldots, x_{t_n} < h\}}\right] - e^{-r_d T} K \mathbb{E}^{\mathbb{Q}_N}\left[\mathbb{1}_{\{x_T > k, x_{t_1} < h, \ldots, x_{t_n} < h\}}\right] \\ &= e^{-r_f T} S_0 \left[\mathbb{Q}_S\left(x_{t_1} < h, \ldots, x_{t_n} < h\right) - \mathbb{Q}_S\left(x_{t_1} < h, \ldots, x_{t_{n-1}} < h, x_{t_n} < k\right)\right] \\ &- e^{-r_d T} K \left[\mathbb{Q}_N\left(x_{t_1} < h, \ldots, x_{t_n} < h\right) - \mathbb{Q}_N\left(x_{t_1} < h, \ldots, x_{t_{n-1}} < h, x_{t_n} < k\right)\right], \end{aligned} \tag{29}$$

using the measures $\mathbb{Q}_N$ and $\mathbb{Q}_S$ as defined in section 3 and the notation $h = \ln H$ and $k = \ln K$.

Again, this formula is independent of the model of the underlying dynamics of S (with respect to an equivalent martingale measure). By choosing a (in)complete model one defines the distribution of S and therefore the values for the probabilities of the events in equation (29). In the Heston model the values for the probabilities in equation (29) can be calculated using the n-variate characteristic functions of section 2. As mentioned in section 2, the evaluation of these probabilities or equivalently these n-multiple integrals can be done by using the result of Shephard's theorem 2 and multidimensional numerical integration. For the calculation of discrete barrier option values, we reformulate theorem 2.

Corollary 1 *Let F denote the distribution function of interest and the integral term $t(\cdot)$ is defined as in equation (21). Assume the requirements of theorem 2 hold. Then*

$$\begin{array}{lcll} 2F(x_1) & = & t(x_1)+1 & \textit{for } n=1 \\ 4F(x_1,x_2) & = & t(x_1,x_2)+t(x_1,0)+t(0,x_2)+1 & \textit{for } n=2 \\ 2^nF(x_1,\ldots,x_n) & = & t(x_1,\ldots,x_n)+\sum_{j_1<\ldots<j_{n-1},0\le j_i\le n} t(x_{j_1},\ldots,x_{j_{n-1}},0) & \\ & & +\ldots+\sum_j t(x_j)+1 & \textit{for } n>2. \end{array}$$

Therefore, for the case of an up-and-out option we need to calculate probabilities of the form $\mathbb{P}(X_1 \le x_1, \ldots, X_n \le x_n) = F(x_1,\ldots,x_n)$ and can use corollary (1) directly.

For the case of a down-and-out option we need to calculate probabilities of the form $\mathbb{P}(X_1 \ge x_1, \ldots, X_n \ge x_n)$. In terms of distribution functions this means to evaluate the terms

$$\mathbb{P}(X_1 \ge x_1,\ldots,X_n \ge x_n) = 1-\sum_{j=1}^n F(x_j)+\sum_{i<j} F(x_i,x_j) \pm \ldots + (-1)^n F(x_1,\ldots,x_n) \quad (30)$$

With corollary 1 this yields

$$\begin{aligned} (30) &= 1-\frac{1}{2}\sum_{j=1}^n (t(x_j)+1)+\frac{1}{4}\sum_{i<j}(t(x_i,x_j)+t(x_i)+t(x_j)+1) \pm \ldots \pm \\ & \quad (-1)^n \frac{1}{2^n}\left(t(x_1,\ldots,x_n)+\sum_{j_1<\ldots<j_{n-1},0\le j_i\le n} t(x_{j_1},\ldots,x_{j_{n-1}})+\ldots+\sum_{j=1}^n t(x_j)+1\right) \\ &= \frac{1}{2^n}\left(1-\sum_{j=1}^n t(x_j)+\sum_{i<j} t(x_i,x_j)\pm\ldots+(-1)^n t(x_1,\ldots,x_n)\right). \end{aligned}$$

Consequently, for the computation of discrete knock-out option values with n fixings, we need to be able to approximate 2^n-1 multi- or one-dimensional integrals numerically. We see that we must find a fast method to calculate n-dimensional distribution functions with respect to their characteristic functions.

In the following section we use and compare this technique with the fast Fourier transform approach, which gives us a general method to compute values of all types of discrete barrier options.

5 Computational Issues

We begin with the implementational aspects for the computation of fader and discrete barrier option values using fast Fourier transform (FFT) methods. We follow the approach of Carr and Madan, established for European-style vanilla options in the one-dimensional case in [5] and the FFT-algorithm of Dempster and Hong in [9] for the correlation option. We describe in detail how to apply the FFT-method for vanilla option valuation to the case of multivariate characteristic functions and thereby the approximation of n-fold integrals. Then we compare the computational results with respect to accuracy and computational times.

5.1 Implementational Aspects of the Fast Fourier Transform Method

In order to evaluate option pricing formulas, such as (23) and (28), we describe a general technique of fast Fourier transforms for options with payoff functions which are dependent on n different spot values in time. As a case study we treat a discrete down-and-out barrier call option with upper barrier level $K < H$,

$$
\begin{aligned}
V_{DOC} &= \mathbb{E}^{\mathbb{Q}_N}\left[e^{-r_d t_n}(S_{t_n} - K)^+ \prod_{i=1}^{n} \mathbb{1}_{\{H \leq S_{t_i}\}}\right] && (31)\\
&= \mathbb{E}^{\mathbb{Q}_N}\left[e^{-r_d t_n}(S_{t_n} - K)\mathbb{1}_{\{H \leq S_{t_1},\ldots,H \leq S_{t_n}\}}\right]. && (32)
\end{aligned}
$$

The above expectation (31) can be calculated in integral form as

$$
E(k,h) = \int_h^\infty \cdots \int_h^\infty e^{-r_d t_n}\left(e^{x_{t_n}} - e^k\right) q(x_{t_1},\ldots,x_{t_n})\, dx_{t_n} \ldots dx_{t_1}, \tag{33}
$$

where the logarithms of strike, barriers and spots K, H, S_{t_i} are denoted by k, h, x_{t_i}. In the case of $H < K$ the lower integration bound of the inner-most integral in (31) would be k instead of h and similarly for UOC options the equivalent of (33) is

$$
\begin{aligned}
&\int_{-\infty}^h \cdots \int_{-\infty}^h e^{-r_d t_n}\left(e^{x_{t_n}} - e^k\right) q(x_{t_1},\ldots,x_{t_n})\, dx_{t_n}\, dx_{t_{n-1}} \ldots\, dx_{t_1} && (34)\\
-&\int_{-\infty}^h \cdots \int_{-\infty}^h \int_{-\infty}^k e^{-r_d t_n}\left(e^{x_{t_n}} - e^k\right) q(x_{t_1},\ldots,x_{t_n})\, dx_{t_n}\, dx_{t_{n-1}} \ldots\, dx_{t_1}. && (35)
\end{aligned}
$$

In the above equations, $\mathbb{Q}_N$ denotes the risk-neutral measure and $q(\cdot)$ the corresponding joint density of the random values x_{t_i}'s for given values x_0 and v_0.

As in [5] and [9], $E(k,h)$ is multiplied by an exponentially decaying term $\exp(\alpha_1 h + \ldots + \alpha_n h)$, for $\alpha_i > 0$, so that it is square-integrable in h over the negative axes. Again, note that for the case of $H < K$ the decaying term of α_n is formed with k instead of h.

The Fourier transform

$$
\psi(v_1,\ldots,v_n) = \int_{\mathbb{R}^n} e^{i(v_1 h + \ldots + v_n h)} e^{\alpha_1 h + \ldots + \alpha_n h} E(k,h)\, dh
$$

of this modified integral can be expressed in terms of the characteristic function φ. The expression for $E(k,h)$ is inserted and the calculation is proceeded similarly as in the one-dimensional case for vanilla options in [5]. Because the characteristic function is known in closed-form, the Fourier transform ψ will also be available analytically in terms of φ. Let $\tilde{v}_j$ denote $v_j - i\alpha_j$, then for $j = 1, \dots, n$ we obtain

- for the down-and-out call with $K < H$

$$\psi(v_1, \dots, v_n) \quad = \quad e^{-r_d t_n} \frac{\varphi(\tilde{v}_1, \dots, \tilde{v}_{n-1}, \tilde{v}_n - i) - e^k \varphi(\tilde{v}_1, \dots, \tilde{v}_n)}{i \prod_{j=1}^n \tilde{v}_j} \tag{36}$$

- for the down-and-out call with $H < K$

$$\psi(v_1, \dots, v_n) \quad = \quad e^{-r_d t_n} \frac{\varphi(\tilde{v}_1, \dots, \tilde{v}_{n-1}, \tilde{v}_n - i)}{(i\tilde{v}_n + 1) i \prod_{j=1}^n \tilde{v}_j}.$$

From the inverse Fourier transform, the integral $E(k,h)$ can be calculated using

$$E(k,h) \quad = \quad \frac{e^{-\sum_{j=1}^n \alpha_j h}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\sum_j v_j h} \psi(v_1, ..., v_n) dv_n \cdots dv_1. \tag{37}$$

For the fader option we can use (37) with $n = 2$. For the first integral term of the value of the up-and-out call in (34) we can use (36) and we can use (37) for the second integral term (35), both with negative arguments in the characteristic function. Furthermore in this case, we choose the dampening parameter such that $\alpha > 1$, and set up the input array of the fast Fourier transform routine as a call of a Fourier transform (not the inverse Fourier transform) of the up-and-out call option, i.e.,

$$(36) \quad = \quad \frac{\exp(\alpha_1 h + \ldots + \alpha_n h)}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(v_1 + \ldots + v_n)h} \psi_1(v_1, \dots, v_n) dv$$

and

$$(37) \quad = \quad \frac{\exp(\alpha_1 h + \ldots + \alpha_n k)}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(v_1 + \ldots + v_{n-1})h + i v_n k} \psi_2(v_1, \dots, v_n) dv$$

with the corresponding Fourier transforms

$$\psi_1(v_1, \dots, v_n) \quad = \quad e^{-r_d t_n} \frac{\varphi(-\tilde{v}_1, \dots, -\tilde{v}_{n-1}, -\tilde{v}_n - i) - e^k \varphi(-\tilde{v}_1, \dots, -\tilde{v}_n)}{i \prod_{j=1}^n \tilde{v}_j},$$

$$\psi_2(v_1, ..., v_n) \quad = \quad -e^{-r_d t_n} \frac{\varphi(-\tilde{v}_1, ..., -\tilde{v}_{n-1}, -\tilde{v}_n - i)}{(i\tilde{v}_n - 1) i \prod_{j=1}^n \tilde{v}_j}.$$

Invoking the trapezoidal rule the Fourier integral in (37) is approximated by the n-fold sum

$$E(k,h) \quad \approx \quad \frac{e^{-\sum_{j=1}^n \alpha_j h}}{(2\pi)^n} \prod_j \Delta_j \underbrace{\sum_{m_1=0}^{N-1} \cdots \sum_{m_n=0}^{N-1} e^{-i\sum_{j=1}^n v_{j,m_j} h} \psi(v_{1,m_1}, \dots, v_{n,m_n})}_{=\Gamma(k)}, \tag{38}$$

where Δ_j denotes the integration step width and

$$v_{j,m_j} = \left(m_j - \frac{1}{2}N\right)\Delta_j \qquad \text{for } m_j = 0, \ldots, N-1.$$

Let the n-fold sums in dependence on the n barrier levels h be denoted by $\Gamma(h, \ldots, h)$.

In order to apply the algorithm of fast Fourier transforms to evaluate the sums in equation (38), we define a grid of size N^n by $\Lambda = \{(h_{1,p_1}, \ldots, h_{n,p_n}) \mid 0 \le p_j \le N-1\}$, where the coordinates are given by

$$h_{j,p_j} = p_j\lambda_j - \frac{1}{2}N\lambda_j + h, \text{ for } j = 1, ..., n.$$

In the case of a down-and-out discrete barrier call with $H < K$, the grid on the last random variable must be $h_{n,p_n} = p_n\lambda_n - \frac{1}{2}N\lambda_n + k$. Choosing $\lambda_1\Delta_1 = \cdots = \lambda_n\Delta_n = \frac{2\pi}{N}$ gives the following values of the n-fold sums $\Gamma(\cdot)$ on the grid Λ as

$$\Gamma(h_{1,p_1}, \ldots, h_{n,p_n}) = \sum_{m_1=0}^{N-1} \cdots \sum_{m_n=0}^{N-1} e^{-i\sum_{j=1}^{n} v_{j,m_j} h_{j,p_j}} \psi(v_{1,m_1}, ..., v_{n,m_n}).$$

This can be computed with the fast Fourier transform by taking the input array as

$$X[m_1, ..., m_n] = (-1)^{\sum_{j=1}^{n} m_j} e^{-i\sum_{j=1}^{n} h\left(m_j\Delta_j - \frac{1}{2}N\Delta_j\right)} \psi(v_{1,m_1}, ..., v_{n,m_n}),$$

such that

$$\Gamma(h_{1,p_1}, \ldots, h_{n,p_n}) = (-1)^{\sum_{j=1}^{n} p_j} \sum_{m_1=0}^{N-1} \cdots \sum_{m_n=0}^{N-1} e^{-\sum_{j=1}^{n} \frac{2\pi i}{N} p_j m_j} X[m_1, ..., m_n]. \tag{39}$$

The result of the FFT algorithm is an output array Y which contains values for the n-folds sums in equation (39) at N^n different logarithmic barrier levels (or logarithmic strike values). The desired approximation of the price of the discrete barrier option is given by the real part of the complex number in Y, which is stored at $Y\left[\frac{1}{2}N, \ldots, \frac{1}{2}N\right]$. It follows that

$$V_{DOC} \approx \frac{e^{-\sum_{j=1}^{n} \alpha_j h}}{(2\pi)^n} (-1)^{\frac{1}{2}nN} \prod_j \Delta_j \times Y\left[\frac{1}{2}N, \ldots, \frac{1}{2}N\right].$$

Remark 7 *Characteristic functions typically have an analytic extension $u \to z \in \mathbb{C}$, regular in some strip parallel to the real z-axis. This aspect plays an import role for the application of fast Fourier transform methods to price options. Hence, to be able to apply the derived n-variate characteristic functions for a numerical analysis of option values within the above FFT methods, we need to make sure during the computations that the expected value $\mathbb{E}[\exp(iux)]$, for $u \in \mathbb{C}$, exists. Lord and Kahl [16] and Lee [22] have analyzed this issue of moment stability for the univariate characteristic function which is used in the closed-form formula for vanilla options in the Heston model.*

5.2 Discussion of Numerical Results

In this section we examine in detail the pricing of fader options and discretely monitored barrier options. We give some examples of sets of model parameters and compare the computation of the pricing values of the above financial products under different numerical methods. Therefore, numerical methods such as Monte Carlo simulation, fast Fourier transform and multidimensional numerical integration are implemented in C# and Mathematica and applied to the described valuation problems.

The stability of the characteristic functions φ^N and φ^S is relevant for the application of the FFT method and also for the numerical integration. As discussed in [1] and [16] there exist two representations of the univariate characteristic function of $\ln S_T$ in the Heston model. Only one of them shows a continuous behavior for all possible model parameters and makes it possible to use implementations of the multi-valued complex logarithm function which calculates only the principal value. Note that, the marginal characteristic functions of $\varphi_{x_{t_1},\dots,x_{t_n}}(u_1,\dots,u_n)$ are continuous as well and can be integrated without a rotation count algorithm due to the results upon the univariate characteristic function in [1]. For the multivariate case the problem of integrating a multi-valued complex logarithm in several dimensions needs still be addressed.

In order to be able to use Monte Carlo simulation with an Euler discretization scheme and to compare the values obtained with the different numerical techniques, the sets of model parameters used in the following sections are especially chosen such that the probability of a negative variance on discrete time grid is low. Nevertheless, the methods using the multivariate characteristic functions are applicable for all combinations of model parameters, if multidimensional integration is applied. To use fast Fourier transform methods, we note that an extension of the multivariate characteristic functions for complex arguments might not be regular for all model parameters.

In the following the problem of pricing fader and discretely monitored barrier options is discussed with regard to the computational accuracy and time between the available pricing methods.

5.2.1 Fader Options

For the comparison of computational accuracy and time we price fade-in calls as stated in (23) with two example sets of model parameters, which are given in table 1. The model parameters were chosen such that the first example represents a market situation with a high speed of mean reversion, a positive correlation and a high variance of volatility value σ_1. Whereas, the second example describes a market, where the volatility of variance σ_2 is lower. The fade-in levels were chosen as a fixed range $[90, 110]$. The time to maturity of the option is set to one year and a monthly fixing. The computational results on the analysis of the accuracy of the different numerical methods to price V_{Fader} are summarized in table 2. The analytic values are calculated with the numerical multidimensional integration functions provided by Mathematica. The Monte Carlo simulations are performed by sampling one million spot paths. They use volatility values, which are observed from the volatility process at 1000 points in time during the lifetime of the option. Additionally, an antithetic variance

Model parameters	κ	θ	ρ	σ_1	σ_2	v_0
	10.0	0.01	0.5	0.2	0.02	0.01
Contract data	K	L	H	T	Fixing	
	100.0	90.0	110.0	1.0	monthly	
Market data	S_0	r_d	r_f			
	100.0	0.05	0.02			

Table 1: Parameter settings for fader option pricing

reduction method is used. The parameters for the FFT are chosen as $N = 512$ integration grid points and $\Delta = 0.3$.

Since the value of the fader option, given in (23) is equal to the sum of fadlets $V_F(t_i)$, for $i = 1, \ldots, 12$, divided by 12, the total accuracy and the total computational time is determined by the corresponding results of each summand. The computational time for the calculation of one summand $V_F(t)$ with Mathematica is approximately 10 minutes, whereas with a Monte Carlo simulation of 1 million sample paths it is about 5 minutes. The calculation with the FFT method requires less than 5 seconds.

In particular, this means by applying the FFT method we are able to compute a value for a fader option with one year maturity and monthly fixings in less than one minute (instead of 2 hours or even 5 minutes). The outputs of all numerical methods yield mostly the same results up to the second decimal place. We can conclude that out of the methods we examined the FFT method is the fastest method, but using a different numerical integration implementation than the one within Mathematica might be more suitable. This thought is followed up in the next section on numerical results of discretely monitored barrier options.

	Example No. 1	Example No. 2
Monte Carlo (1 million sample paths)	3.4004	3.5127
0.975-confidence interval	(3.3972,3.4036)	(3.5099,3.5155)
Numerical Integration (with Mathematica)	3.4012	3.5152
Fast Fourier Transform	3.4013	3.5153

Table 2: Numerical results for fader call option values in the Heston model.

5.2.2 Discrete Barrier Options

The analysis of the valuation of discretely monitored barrier options is initiated by a comparison of the available pricing methods with respect to accuracy and computational time. We work with the example settings in table 3. We use the following methods for the calculation of one value of a

	Model parameters					Market data			Contract data		
Down-and-Out	κ	σ	θ	ρ	v_0	r_d	r_f	S_0	H	K	t_n
	5.0	0.5	0.1	0.5	0.5	0.05	0.0	100.0	95.0	90.0	1.0
Up-and-Out	κ	σ	θ	ρ	v_0	r_d	r_f	S_0	H	K	t_n
	5.0	0.1	0.1	0.5	0.1	0.05	0.02	100.0	120.0	80.0	1.0

Table 3: Example set of model parameters for the pricing of down-and-out and up-and-out discrete barrier options.

particular discrete barrier option:

- Monte Carlo simulation with 1 million and 10 million spot paths, respectively. For the discretization of the time horizon of the volatility process an Euler scheme and 1000 steps were chosen. This discretization of the variance process is fine enough for this example to ensure that the process attains mostly non-negative values. An antithetic variance reduction method was applied.

- Fast Fourier transform methods as described in the previous section. The parameters of the FFT method were set to values between $N = 2^4$ and $N = 2^7$, the discretization grid for the numerical integration was chosen equally for every dimension, i.e. $\Delta = 0.5$ in the down-and-out case and $\Delta = 0.3$ in the up-and-out case.

- Multidimensional numerical integration. The multidimensional integral is estimated using a Romberg integration method which is based on the midpoint rule. The number of subintervals into which the i-th integration interval is initially subdivided is set to 30 for the case of up-and-out calls and to 60 for down-and-out calls. This integration technique was developed by Davis and Rabinowitz in [7]. The C++ version of this integration method can be found in [30].

We illustrate values of discrete down-and-out barrier options with different numbers of fixings n, for $n = 1, \ldots, 6$,

$$(S_{t_n} - K)^+ \prod_{i=1}^{n} \mathbb{1}_{\{S_{t_i} \geq H\}},$$

where the fixing times $t_i = \left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$ are chosen equidistant from each other. The results with Monte Carlo simulation, numerical integration and FFT are listed in table 4. For the comparison of computational time and accuracy, the method of multidimensional numerical integration is applied with respect to formula (30).

We observe that the values for the down-and-out barrier options with fixings up to four lie close together for all of the three numerical methods. The values which are computed with FFT and the numerical integration lie in the 97.5% confidence intervals of the Monte Carlo simulations. The values of the Monte Carlo simulation with 10 million simulated spot paths and the values of the other two methods coincide up to the first decimal place, which is equivalent to an accuracy of one-tenth of a percent of the underlying. The same accuracy could not be achieved for barrier options with more than four fixings, which is a result of the low number of grid points N used in the FFT method. We note that we used the FFT routine of the Numerical Recipes [26], which requires a one-dimensional input array of size $2 \cdot N^{\#\text{fixings}}$. Due to memory capacity, this limits the number of grid points N for the case of 5 and 6 fixings to $N = 32$ and $N = 16$, respectively. Hence, N can only be increased, if the FFT method is called multiple times for different integration regions. Consequently, by dividing the calls of the FFT routine up into several single calls, the deviation of the values between Monte Carlo and FFT could be corrected.

However, this technique and the increase of number of grid points increases the computational time of the FFT algorithm. The computation of values with Monte Carlo simulation with 1 million spot paths takes about 5 minutes for each option and about 50 minutes if 10 million spot paths are generated and evaluated. The computational times for the FFT method are 1 and 45 seconds, 8, 20 and 11 minutes for barrier options with 2 up to 6 fixings, respectively. The multidimensional numerical integration routine is not limited to a certain number of grid points and therefore the values computed with this method result in a higher accuracy than the results obtained with FFT, but as expected also requires a much higher computational time. It takes 1 second for a down-and-out call with 2 fixings, 3 minutes for 3 fixings, 17 hours for 4 fixings and several days for 5 and 6 fixings.

For the calculation of values of up-and-out barrier options with fixings n, for $n = 1, \dots, 6$,

$$(S_{t_n} - K)^+ \prod_{i=1}^{n} \mathbb{1}_{\{S_{t_i} \leq H\}}, \qquad \text{for} \quad t_i = \left\{ \frac{1}{n}, \frac{2}{n}, \dots, 1 \right\},$$

the techniques of multidimensional numerical integration uses formula (29) and the result of corollary 1. Basically, the numerical integration uses the multivariate characteristic functions given in equation (6) and (15). The fast Fourier transform method depends on the same functions, but extended to complex arguments. So, the comparison of the two methods mainly lies in the comparison of the computational time. All the results for the three numerical methods are listed in table 5.

The computational time of the FFT routine for up-and-out barrier options doubles compared to the down-and-out case, since here the FFT routine has to be called twice, because of (35). However, for the multidimensional numerical integration routine the computational times reduce as the overall accuracy can already be achieved with half of the number of initial subintervals than the one used for the down-and-out barrier case. The computation of an up-and-out call value with 2 fixings takes only 1 second, 1.3 minutes for 3 fixings, but 1 hour and 2 days for barrier call values with 4 and 5 fixings.

In the one-dimensional case the computational time of the FFT routine to compute vanilla option values compared to numerical integration with certain caching techniques is higher as shown in [19].

Number of fixings	Monte Carlo 1 million paths (0.975-confidence)	Monte Carlo 10 million paths (0.975-confidence)	Fast Fourier Transform N	Multidimensional numerical integration subdivisions
2	21.4671	21.4326	21.4483	21.4447
	(21.3889,21.5454)	(21.4079,21.4574)	256	60
	5 min	50 min	1 s	1 s
3	20.0954	20.1234	20.1146	20.1095
	(20.0177,20.1732)	(20.0987,20.1480)	128	60
	5 min	50 min	45 s	3 min
4	19.1284	19.1151	19.1241	19.1095
	(19.0510,19.2058)	(19.0906,19.1395)	64	60
	5 min	50 min	8 min	17 h
5	18.3995	18.3432	18.6877	18.2456
	(18.3224,18.4766)	(18.3189,18.3675)	32	30
	5 min	50 min	20 min	too long
6	17.7225	17.6827	16.3318	7.0305
	(17.6461,17.7988)	(17.6586,17.7069)	16	10
	5 min	50 min	11 min	too long

Table 4: Values for a discretely monitored down-and-out barrier option with parameters given in table 3 for three different numerical methods.

However, in the multivariate case our examples show that the choice between the various numerical methods (without caching techniques) is not such a clear-cut decision[4].

Number of fixings	Monte Carlo 1 million paths (0.975-confidence)	Monte Carlo 10 million paths (0.975-confidence)	Fast Fourier Transform N	Multidimensional numerical integration subdivisions
2	7.0188	7.0228	7.0217	7.0217
	(7.0028,7.0349)	(7.0177,7.0279)	256	30
	5 min	50 min	2 s	1 s
3	6.3238	6.3221	6.3339	6.32403
	(6.3085,6.3391)	(6.3172,6.3269)	128	30
	5 min	50 min	1.3 min	1 min
4	5.8887	5.8884	5.6900	5.8931
	(5.8738,5.9035)	(5.8837,5.8931)	64	30
	5 min	50 min	16 min	3 h
5	5.5930	5.5972	1.8902	5.2625
	(5.5785,5.6074)	(5.5927,5.6018)	32	30
	5 min	50 min	22 min	2d

Table 5: Values for a discretely monitored up-and-out barrier option with parameters given in table 3 for three different numerical methods. We have used $\alpha = 1.75$.

6 Summary

We have shown how to compute values of faders and discretely monitored barrier options in the Heston model in closed-form by extending the valuation method using multiple Fourier transforms. The resulting characteristic function of a vector of logarithmic spot prices can be computed explicitly using a recursion. The methodology presented extends to other stochastic volatility models. The important property turns out to be a known characteristic function. We have also demonstrated that our results can be used in practice. We have benchmarked and verified our closed-form solutions in a multidimensional integration and an FFT method against Monte Carlo and are able to speed up the computation significantly if the number of fixings is small.

References

[1] Albecher, H., Mayer, P., Schoutens, W. and Tistaert, J. (2007). *The Little Heston Trap*. *Wilmott* No.1, 83–92.

[4] Note that when evaluating options with different strike and barrier levels we need not to recompute the characteristic function when evaluating the integrals in (30)

[2] ANDERSEN, L. (2007). *Efficient Simulation of the Heston Stochastic Volatility Model*. Bank of America Working paper, available at http://ssrn.com/abstract=946405.

[3] BECKER, C. and WYSTUP, U. (2005). *On The Cost of Delayed Fixing Announcements in FX Options Markets*. Research Report, Frankfurt School of Finance & Management, Center for Practical Quantitative Finance.

[4] BROADIE, M. and KAYA, Ö. (2006). *Exact Simulation of Stochastic Volatility and other Affine Jump Diffusion Models*. Operations Research. Vol. 54, No. 2, 217–231.

[5] CARR, P. and MADAN, D. (1999). *Option Valuation Using the Fast Fourier Transform*. Journal of Computational Finance. Vol. 2 No. 4, 61–73.

[6] CHIARELLA C., ZIOGAS A. (2006). *Pricing American Options under Stochastic Volatility*. Working Paper.

[7] DAVIS, P. and RABINOWITZ, P. (1984). *Methods of Numerical Integration*. Dover Publications.

[8] DA FONSECA, J., GRASSELLI, M. and TEBALDI, C. (2005). *Wishart Multi-Dimensional Stochastic Volatility*. Working Paper.

[9] DEMPSTER, M. and HONG, S. (2000). *Spread Option Valuation and the Fast Fourier Transform*. Working Paper. University of Cambridge.

[10] DUFFIE, D., SINGLETON, K. and PAN, J. (2000). *Transform Analysis and Asset Pricing for Affine Jump-Diffusions*. Econometrica. Vol. 68, 1343–1376.

[11] FAULHABER, O. (2002). *Analytic Methods for Pricing Double Barrier Options in the Presence of Stochastic Volatility*. Master Thesis.

[12] GRÜNBICHLER, A., and LONGSTAFF, F., 1996. *Valuing Futures and Options on Volatility*. Journal of Banking and Finance 20, 985–1001.

[13] HAKALA, J. and WYSTUP, U. (2002). *Foreign Exchange Risk*. Risk Publications, London.

[14] HESTON, S.L. (1993). *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*. Review of Financial Studies 6, 327–343.

[15] HIGHAM, D. and MAO, X. (2005). *Convergence of Monte Carlo Simulations Involving the Mean-Reverting Square Root Process*. Journal of Computational Finance, Vol.8 No.3, 35–62.

[16] KAHL, C. and LORD, R. (2006). *Why the Rotation Count Algorithm Works*. Working Paper.

[17] C. KAHL & R. LORD (2006). *Optimal Fourier Inversion in Semi-Analytical Option Pricing*. Working paper.

[18] C. Kahl & P. Jäckel (2005). *Not so Complex Logarithms in the Heston Model*. Working paper.

[19] Kilin, F. (2007). *Accelerating the Calibration of Stochastic Volatility Models*. Research Report No. 6, Frankfurt School of Finance & Management, Center for Practical Quantitative Finance.

[20] Kluge, T. (2003). *Pricing Derivatives in Stochastic Volatility Models using the Finite Difference Method*. Diploma Thesis. Technical University of Chemnitz.

[21] Kruse, S. and Nögel, U. (2005). *On the Pricing of Forward Starting Options in Hestons Model on Stochastic Volatility*. Finance and Stochastics 9, 233–250.

[22] Lee, R. W. (2004). *Option Pricing by Transform Methods: Extensions, Unification and Error Control*. Journal of Computational Finance Vol.7 No.3,51–86.

[23] Lord, R., Koekkoek, R. and van Dijk, D. (2006). *A Comparison of Biased Simulation Schemes for Stochastic Volatility Models*. Working Paper.

[24] Lukacs, E. and Laha, R. G. (1964). *Applications of Characteristic Functions*. Lubrecht & Cramer Ltd.

[25] Overhaus, M. et al. (1999). *Modelling and Hedging Equity Derivatives*. Risk Books.

[26] Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T. (1993). *Numerical Recipes in C*. Cambridge University Press.

[27] Shephard, N. G. (1991). *From Characteristic Function to Distribution Function: A Simple Framework for the Theory*. Econ. Theory. Cambridge University Press, Vol. 7(4), 519–29.

[28] Wystup, U. (2006). *FX Options and Structured Products*. Wiley Finance.

[29] Zhu, J. (2000). *Modular Pricing of Options - An Application of Fourier Analysis*. Lecture Notes in Economics and Mathematical Systems 493. Berlin/Heidelberg: Springer-Verlag.

[30] *Numerical integration source code in C++*.
http://people.scs.fsu.edu/~burkardt/cpp_src/nintlib/nintlib.C