FX Smile Modelling

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Contents

Abstract

This paper provides a short introduction into the handling of FX implied volatility market data - especially their inter- and extrapolation across delta space and time. We discuss a low-dimensional Gaussian kernel approach as method of choice showing several advantages over usual smile interpolation methods like e.g. cubic splines.

1 FX Implied Volatility

Implied volatilities for FX Vanilla options are normally quoted against Black Scholes deltas. Note that these deltas already process the volatility to be quoted which makes iterative processes necessary to determine unique strike-volatility coordinates. However, under normal circumstances the mapping works via a quickly converging fixed point iteration.

Proposition 1.1 (Delta-Strike Fixed Point Iteration)
Let

$$\Delta_n : A \mapsto A,\ \ A \subset (0,1)\ \textit{be a mapping, defined by}$$

$$\begin{aligned} \sigma_0 &= \sigma_{ATM}, \\ \Delta_0 &= \Delta(K_{Call}, \sigma_{ATM}) \end{aligned}$$

$$\begin{aligned} \Delta_{n+1} &= e^{-r_f(T-t)} N(d_1(\Delta_n)) \\ &= e^{-r_f(T-t)} N\left(\frac{\ln(S/K) + (r_d - r_f + \frac{\sigma^2(\Delta_n)}{2})(T-t)}{\sigma(\Delta_n)\sqrt{T-t}}\right) \end{aligned}$$

For sufficiently large $\sigma(\Delta_n)$ and a smooth, differentiable volatility smile the sequence converges for $n \to \infty$ against the unique fixed point $\Delta^* \in A$ with $\sigma^* = \sigma(\Delta^*)$, corresponding to strike K.

Usual FX smiles normally satisfy the above mentioned regularity conditions. More details concerning this proposition can be found in Wystup [5]. However note, that already here smoothness is demanded - which directly leads to the issue of an appropriate smile interpolation.

2 Interpolation

Before the discussion of specific interpolation methods it is recommended to take a step backwards and remember Rebonato‘s well-known statement of implied volatility as the wrong number in the wrong formula to obtain the right price [3]. So the explanatory power of implied volatilities for the dynamics of a stochastic process remains limited. Implied volatilities give a lattice on which marginal distributions can be constructed. However, even using many data points to generate marginal distributions, forward distributions and extremal distributions - determining the prices of e.g. compound and barrier products - cannot be uniquely defined by implied volatilities (see Tistaert et al.[4] for a discussion of this.

The attempt to capture FX smile features can lead into two different general approaches.

2.1 Parametrisation

One possibility to express smile or skew patterns is just to capture it as the calibration parameter set of an arbitrary stochastic volatility or jump diffusion model which generates the observed market implied volatilities. However, as spreads are rather narrow in liquid FX options markets, it is preferred to exactly fit the given input volatilities. This automatically leads to an interpolation approach.

2.2 Pure Interpolation

As an introduction we would like to pose four requirements for an acceptable volatility surface interpolation:

1. Smoothness in the sense of continuous differentiability. Especially with respect to the possible application of Dupire-style local volatility models it is crucial to construct an interpolation which is at least $\mathcal{C}^2$. This becomes obvious when looking at the expression for the local volatility in this context:

$$\sigma_t^{local}(S(t)) = \left(2 \frac{\frac{\partial Call(S,t;K,T)}{\partial T} + \frac{\partial Call(S,t;K,T)}{\partial K}}{K^2 \frac{\partial^2 Call(S,t;K,T)}{\partial K^2}} \right)^{\frac{1}{2}}.$$

   Note in addition that local volatilities can directly be extracted from delta-based FX volatility surfaces, i.e. the Dupire formula can alternatively be expressed in terms of delta. See Hakala and Wystup [2] for details.

2. Absence of oscillations, which is guaranteed if the sign of the curvature of the surface does not change over different strike or delta levels

3. Absence of arbitrage possibilities on single smiles of the surface as well as absence of calendar arbitrage

4. A reasonable extrapolation available for the interpolation method

A classical interpolation method widely spread are cubic splines. They attempt to fit surfaces by fitting piecewise cubic polynomials to given data points. They are specified by matching their second derivatives at each intersection. While this ensures the required smoothness by construction, it does not prevent oscillations - which directly leads to the danger of arbitrage possibilities - nor does it define how to extrapolate the smile. We therefore introduce the concept of a slice kernel volatility surface - as described in Hakala and Wystup [2] - as an alternative:

Definition 2.1 (Slice Kernel) *Let $(x_1, y_1), (x_2, y_2)..., (x_n, y_n)$ be n given points and $g : \mathbb{R} \mapsto \mathbb{R}$ a smooth function which fulfils*

$$g(x_n) = y_n, \ \forall n = 1, ..., n.$$

A smooth interpolation is then given by

$$g(x) := \frac{1}{\Phi_\lambda(x)} \sum_{i=1}^{N} \alpha_i K_\lambda(\|x - x_i\|),$$

where

$$\Phi_\lambda(x) := \sum_{i=1}^{N} K_\lambda(\|x - x_i\|)$$

and

$$K_\lambda(u) := \exp\left\{-\frac{u^2}{2\lambda^2}\right\}.$$

The described kernel is also called Gaussian Kernel. The interpolation reduces to the determination of the α_i which is straightforward via solving a linear equation system. Note that λ remains as free smoothing parameter which also impacts the condition of the equation system. At the same time it can be used to fine-tune the extrapolation behavior of the kernel.

Normally the slice kernel produces reasonable output smiles based on a maximum of seven delta-volatility points. Then it fulfills all above mentioned requirements: It is $\mathcal{C}^\infty$, does not create oscillations, passes typical no-arbitrage conditions as they are e.g. posed by Gatheral [1], and finally has an inherent extrapolation method.

In time direction one might connect different slice kernels by linear interpolation of the variances for same deltas. This also normally ensures the absence of calendar arbitrage, for which a necessary condition is a non-decreasing variance for constant moneyness F/K (see also Gatheral [1] for a discussion of this).

Figure 1 below displays the shape of a slice kernel applied to a typical FX vol surface constructed from 10 and 25 delta volatilities, and the ATM volatility.

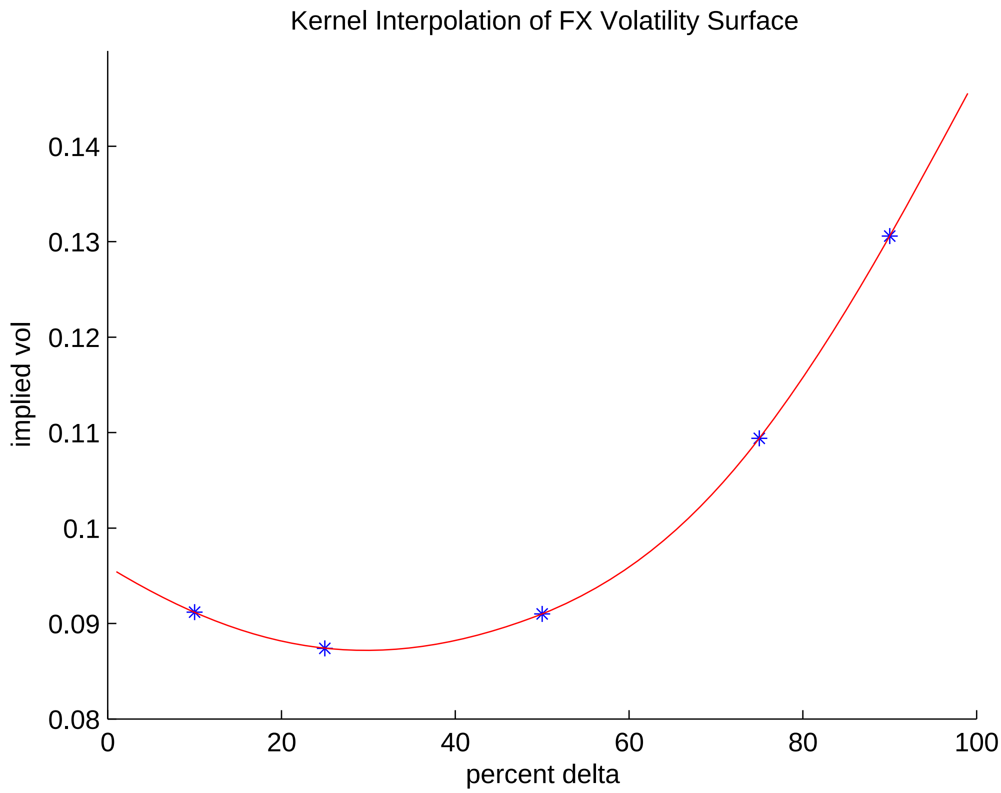

Figure 1: Kernel Interpolation of FX Vol Surface

References

[1] Gatheral, J., A Parsimonious Arbitrage-Free Volatility Parametrization with Application to the Valuation of Volatility Derivatives, Presentation, Global Derivatives and Risk Management Conference, May 2004

[2] Hakala, J., Wystup, U., Local Volatility Surfaces - Tackling The Smile, published in Foreign Exchange Risk, Risk Books, 2002

[3] Rebonato, R., Volatility and Correlation, Wiley, 1999

[4] Tistaert, J., Schoutens, W., Simons, E., A Perfect Calibration! Now What?, Wilmott Magazine, March 2004

[5] Wystup, U., FX Options and Structured Products, Wiley, 2007