Pricing of First Generation Exotics with the Vanna-Volga Method: Pros and Cons

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Professional

- Diplom in Mathematics, Goethe University Frankfurt
- Ph.D. in Mathematical Finance, Carnegie Mellon University, Pittsburgh
- Professor of Quantitative Finance at Frankfurt School since Oct 2003

- 10 Years of Trading Floor experience at Citibank, UBS, Sal. Oppenheim, Commerzbank as Quant and Structurer

- Expert in FX Options -- Training, Consulting and Software Production for the Financial Industry (MathFinance AG)

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Personal

- Married since 1993, two children, one dog

- Lives in Waldems (Taunus)

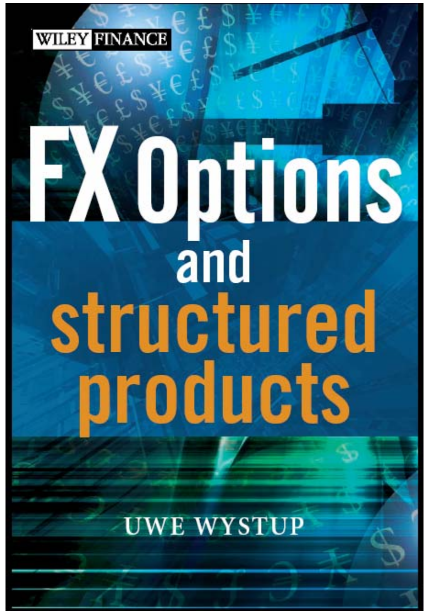
- Playing for church services in Steinfischbach, Wallrabenstein, Mauloff, Wüstems

- Enjoys Biking, Hiking, Swimming an Golf

Selected Publications

Jürgen Hakala and Uwe Wystup
Foreign Exchange Risk
Risk Publications, London 2002
http://www.mathfinance.com/FXRiskBook/

Uwe Wystup
FX Options and Structured Products
Wiley Finance, 2006
http://fxoptions.mathfinance.com/



Efficient computation of option price sensitivities using homogeneity and other tricks, joint with Oliver Reiss, *The Journal of Derivatives* Vol. 9 No. 2, Winter 2001

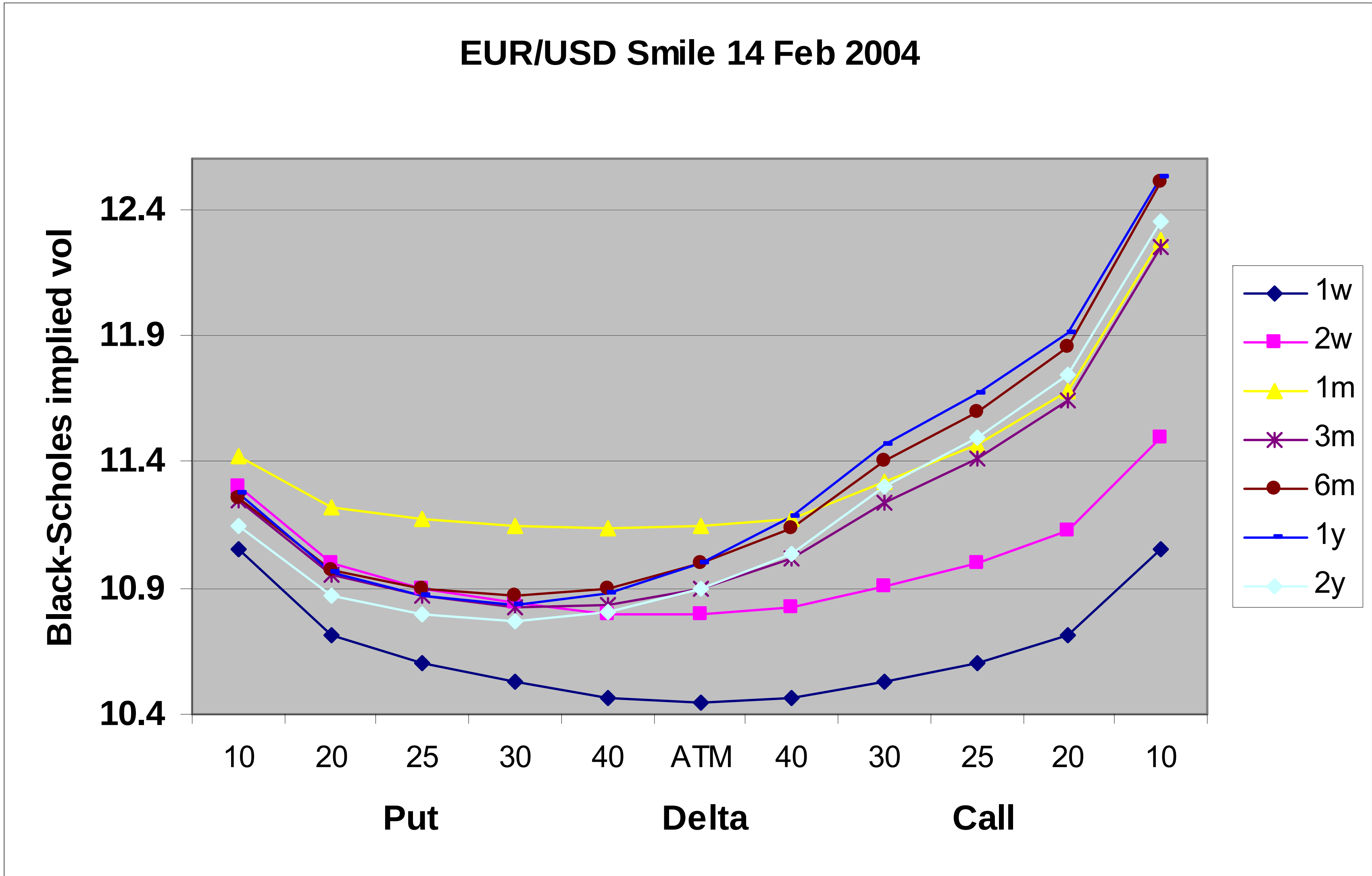
Valuation of exotic options under short selling constraints, joint with Steven E. Shreve and Uwe Schmock, *Finance and Stochastics* VI, 2 (2002)

The market price of one-touch options in foreign exchange markets, *Derivatives Week* Vol. XII, no. 13, London 2003

Efficient Computation of Option Price Sensitivities for Options of American Style, joint with Christian Wallner, *Wilmott*. 2004

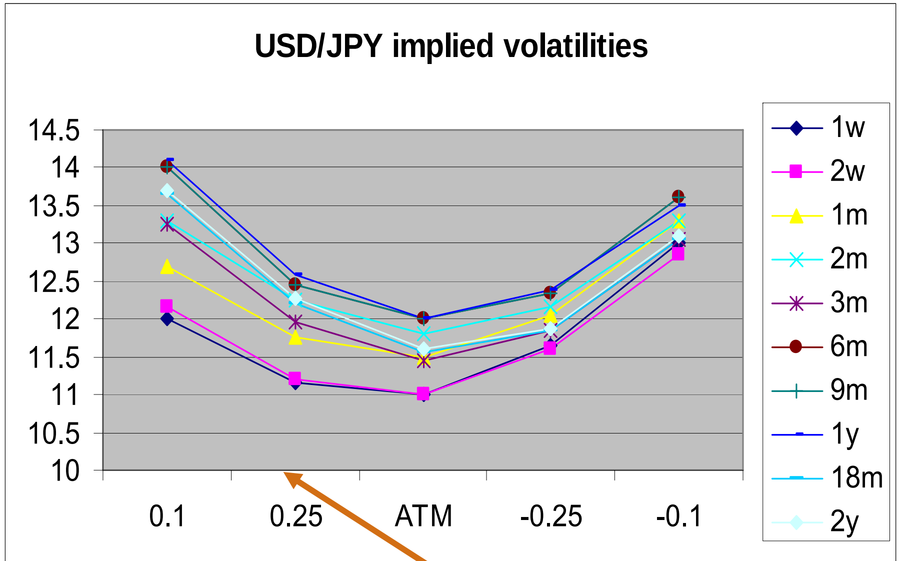
The Heston Model and the Smile, joint with Rafal Weron, Chapter contribution to the book *Statistical Tools for Finance and Insurance (STF)*, eds. Pavel Cizek, Wolfgang Haerdle, Rafal Weron. 2004

Result from the Market: Smile Effect in the Black-Scholes/Merton Model



What are Smile and Skew ?

On the axes:



y-axis: implied volatility (in %)

x-axis (for equity)

Strikes

x-axis for FX

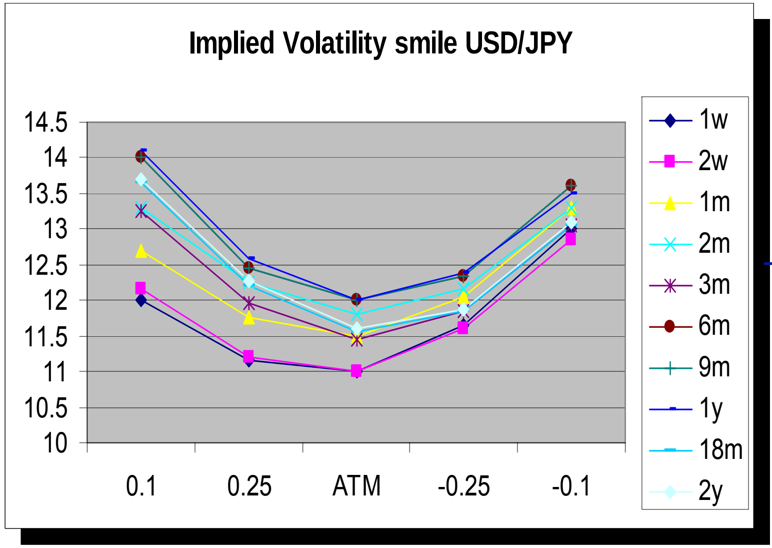
Delta of call options (10% and 25%), at-the-money (ATM), Delta of put options (-25% and -10%)

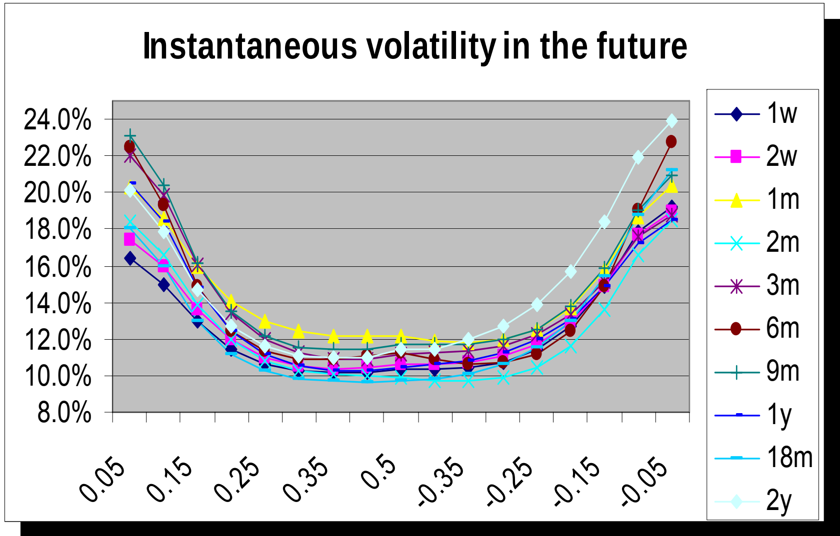
Volatility Smile for Vanilla Options – Model History

- **Black-Scholes: geometric Brownian motion: σ constant** $$dS_t = (r_d - r_f) S_t \, dt + \sigma S_t \, dW_t$$
- **Black-Scholes with time-dependent parameters** $$\sigma = \sigma(t)$$
- **Deterministic (*local-vol*) Model** $$\sigma = \sigma(S_t, t)$$
- **Stochastic Volatility** σ Stochastic Process
- **Uniform Volatility Model (UVM)** "Universal Barriers", Risk May 2002, Alexander Lipton & William McGhee

Is the Smile for Vanilla Options deterministic?

- **Instantaneous Volatility for the Spot price process depends on:**
  - **Time to maturity**
  - **Spot**
- **and is completely determined by today's smile**





Real Time Volatility Smile Surfaces on Reuters

Quote: EURVOLSURF

EURVOLSURF EUR VOLSURFACE

Tue 18 Dec 2007 13:24 GMT Standard Time Logical Displays<0#EURVOL=R>

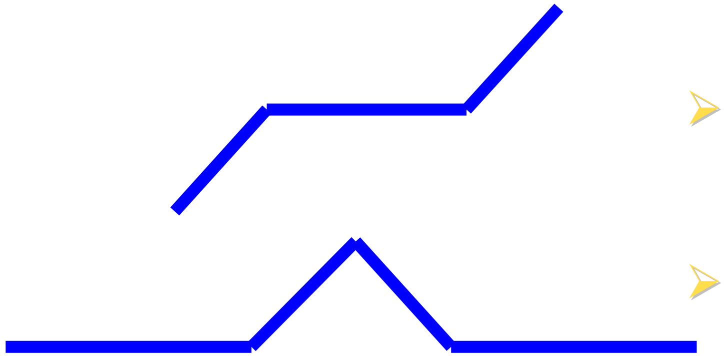
	10DPUT	15DPUT	20DPUT	25DPUT	30DPUT	35DPUT	40DPUT	45DPUT	ATM	45DCALL	40DCALL
1W	11.150	11.056	10.951	10.825	10.671	10.499	10.324	10.161	10.025	9.926	9.863
2W	10.975	10.840	10.704	10.563	10.416	10.269	10.126	9.993	9.875	9.778	9.706
3W	10.763	10.574	10.393	10.225	10.075	9.943	9.826	9.721	9.625	9.538	9.469
1M	10.825	10.460	10.122	9.838	9.625	9.474	9.365	9.280	9.200	9.113	9.038
6W	10.826	10.373	9.956	9.611	9.364	9.199	9.090	9.010	8.934	8.845	8.766
2M	10.600	10.207	9.843	9.537	9.312	9.153	9.041	8.955	8.875	8.788	8.714
3M	10.725	10.303	9.914	9.588	9.348	9.182	9.066	8.980	8.900	8.813	8.739
4M	10.662	10.240	9.848	9.518	9.271	9.097	8.973	8.881	8.799	8.715	8.647
5M	10.365	9.999	9.656	9.360	9.128	8.952	8.819	8.716	8.628	8.549	8.492
6M	10.075	9.756	9.455	9.188	8.968	8.793	8.653	8.541	8.450	8.374	8.326
9M	10.500	9.993	9.526	9.138	8.857	8.668	8.543	8.454	8.375	8.287	8.216
1Y	10.525	10.001	9.517	9.112	8.817	8.614	8.478	8.382	8.300	8.215	8.151
2Y	10.475	9.930	9.428	9.013	8.715	8.518	8.394	8.314	8.250	8.183	8.135

Implied Volatility in the Options Market

EUR/GBP	Spot Rate	Option Volatility						25 Delta Risk Reversal			25 Delta Strangle		
Date:		1 Week	1 Month	3 Month	6 Month	1 Year	2 Years	1 Month	3 Month	1 Year	1 Month	3 Month	1 Year
01. Apr 05	0,6864	4,69	4,83	5,42	5,79	6,02	6,09	0,18	0,23	0,30	0,15	0,16	0,16
04. Apr 05	0,6851	4,51	4,88	5,34	5,72	5,99	6,07	0,15	0,20	0,29	0,15	0,16	0,16
05. Apr 05	0,6840	4,66	4,95	5,34	5,70	5,97	6,03	0,11	0,19	0,28	0,15	0,16	0,16
06. Apr 05	0,6847	4,65	4,91	5,39	5,79	6,05	6,12	0,08	0,19	0,28	0,15	0,16	0,16
07. Apr 05	0,6875	4,78	4,97	5,39	5,79	6,01	6,10	0,13	0,19	0,28	0,15	0,16	0,16
08. Apr 05	0,6858	4,76	5,00	5,41	5,78	6,00	6,09	0,13	0,19	0,28	0,15	0,16	0,16
11. Apr 05	0,6855	4,69	5,12	5,44	5,77	6,03	6,12	0,12	0,19	0,28	0,15	0,16	0,16
12. Apr 05	0,6832	4,75	5,15	5,48	5,79	6,02	6,11	0,12	0,18	0,28	0,15	0,16	0,16
13. Apr 05	0,6814	4,70	5,21	5,52	5,83	6,09	6,15	NA	0,18	0,28	0,15	0,16	0,16
14. Apr 05	0,6808	4,71	5,16	5,59	5,84	6,10	6,13	0,11	0,18	0,28	0,15	0,16	0,16
15. Apr 05	0,6816	4,78	5,07	5,48	5,77	6,00	6,09	0,09	0,17	0,28	0,15	0,16	0,16
18. Apr 05	0,6833	4,98	5,30	5,66	5,96	6,19	6,20	0,12	0,19	0,28	0,15	0,16	0,16
19. Apr 05	0,6805	5,25	5,43	5,78	6,08	6,29	6,29	0,12	0,20	0,29	0,15	0,16	0,16
20. Apr 05	0,6815	5,37	5,54	5,88	6,14	6,34	6,32	0,10	0,20	0,29	0,15	0,16	0,16
21. Apr 05	0,6841	5,30	5,50	5,87	6,11	6,31	6,31	0,08	0,20	0,29	0,15	0,16	0,16
22. Apr 05	0,6826	4,86	5,36	5,71	5,98	6,18	6,20	0,08	0,20	0,29	0,15	0,16	0,16
25. Apr 05	0,6791	4,92	5,47	5,73	6,01	6,20	6,26	0,10	0,18	0,29	0,15	0,16	0,16
26. Apr 05	0,6808	4,50	5,32	5,62	5,94	6,12	6,20	0,06	0,17	0,28	0,14	0,15	0,16
27. Apr 05	0,6797	4,73	5,33	5,65	5,95	6,14	6,22	0,05	0,16	0,27	0,14	0,15	0,16
28. Apr 05	0,6766	5,12	5,41	5,65	5,99	6,18	6,23	0,04	0,13	0,24	0,15	0,16	0,16
29. Apr 05	0,6766	5,28	5,40	5,70	6,01	6,19	6,25	0,05	0,13	0,25	0,15	0,16	0,16

Source: BBA (British Bankers' Association) http://www.bba.org.uk

Butterfly and Risk Reversal



OTM put - ATM put - ATM call + OTM call

- Risk Reversal: long call + short put
- Butterfly consists of 4 Vanilla Options

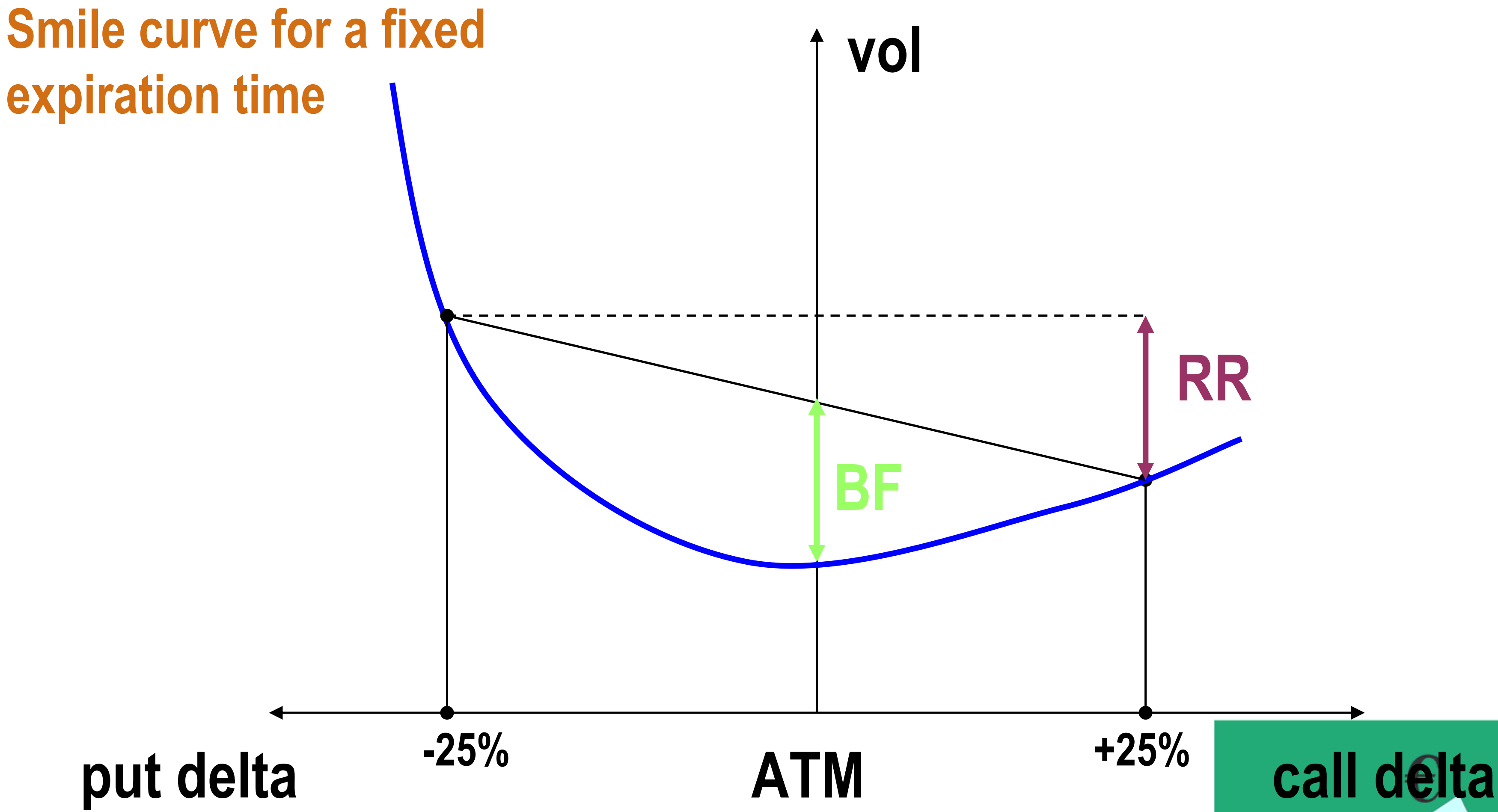
→ BF = ½(OTMcallvol + OTMputvol) - ATMvol

0.8 = ½(9.8 + 10.2) - 9.2

→ RR = OTMcallvol - OTMputvol

-0.4 = 9.8 - 10.2

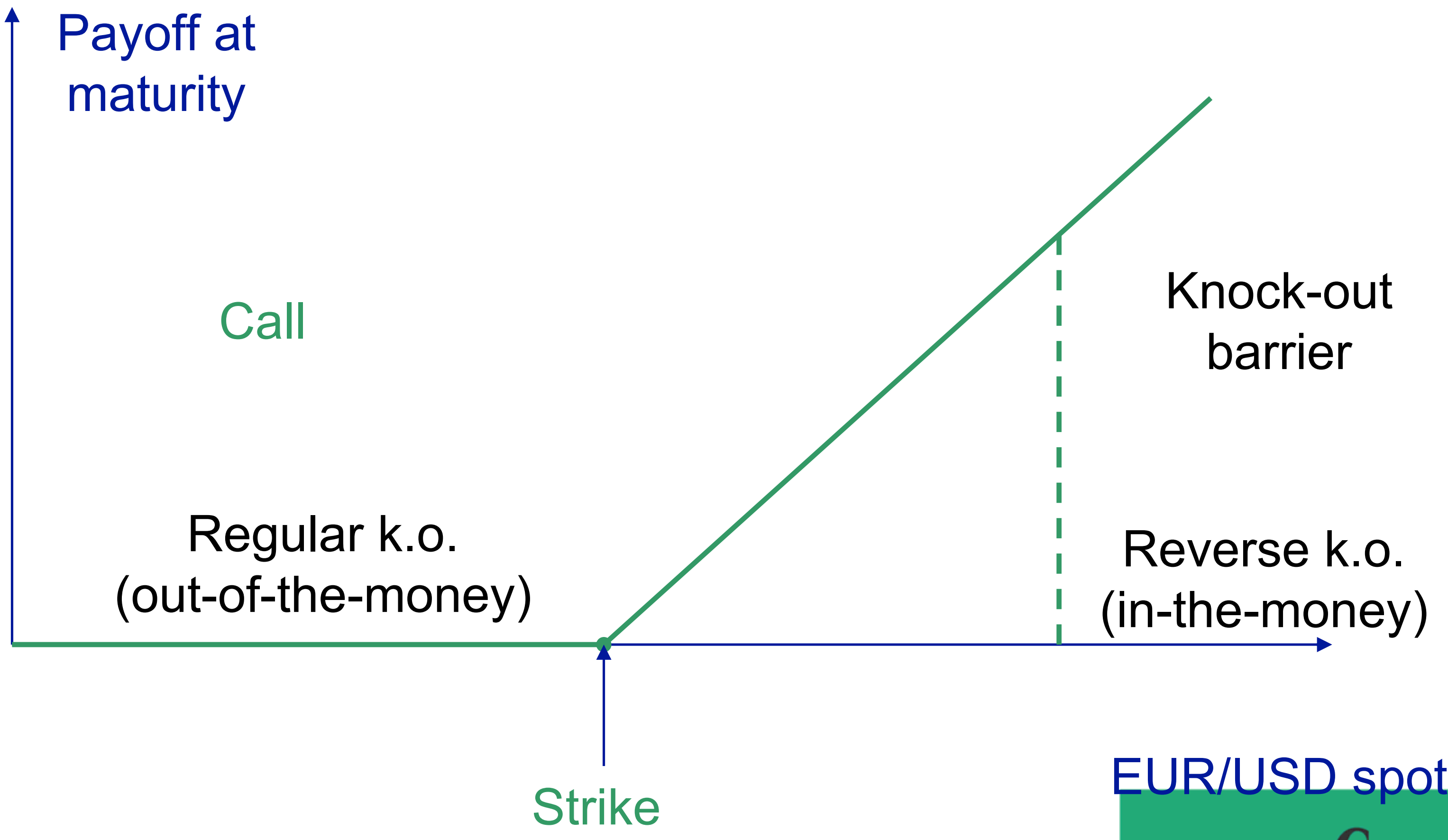
Butterfly and Risk Reversal



First Generation Exotics

- **Digitals, Barriers, Touches**
  - KO, KI, RKO, RKI, DKO, DKI
  - OT, NT, DOT, DNT
  - Digitals, EKO, KIKO, TA
- **Compound and Instalments**
- **Asian**
  - Fixed Strike and Floating Strike
- **Others**
  - Lookback
  - Power
  - Chooser
  - Quanto

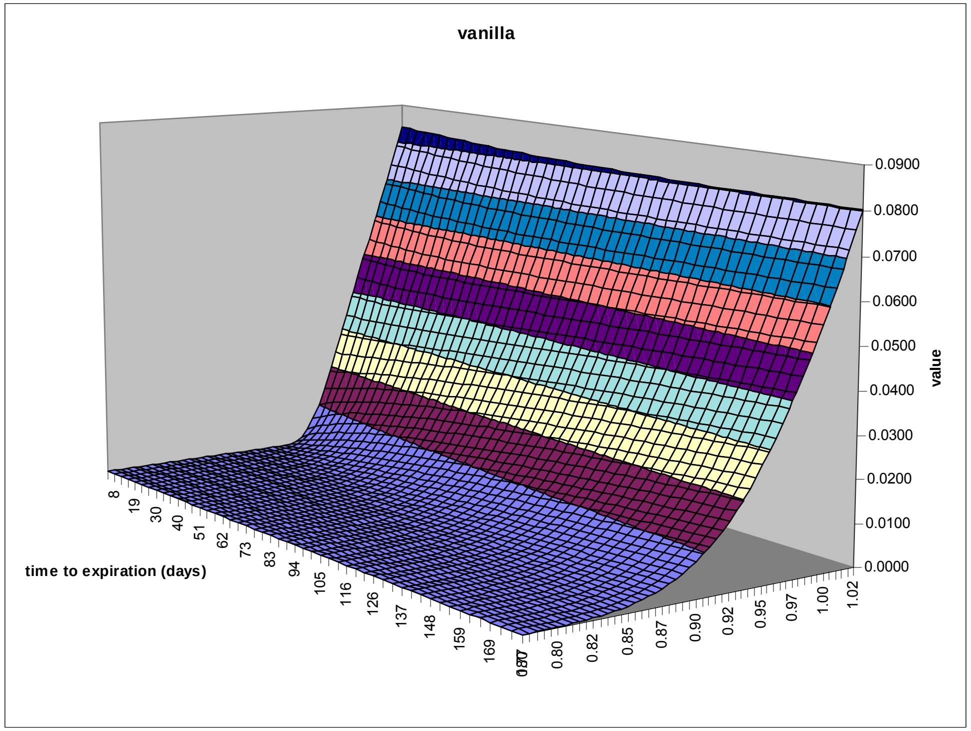
Barrier Options Terminology

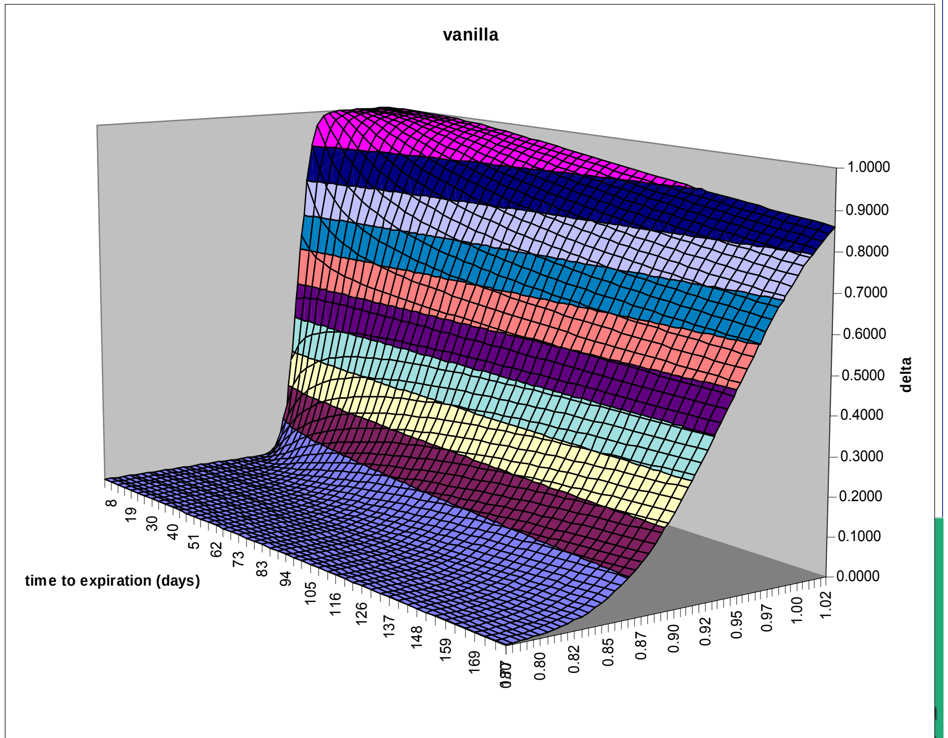


- **Barrier levels are valid at *all* times**

Hedging Barrier Options

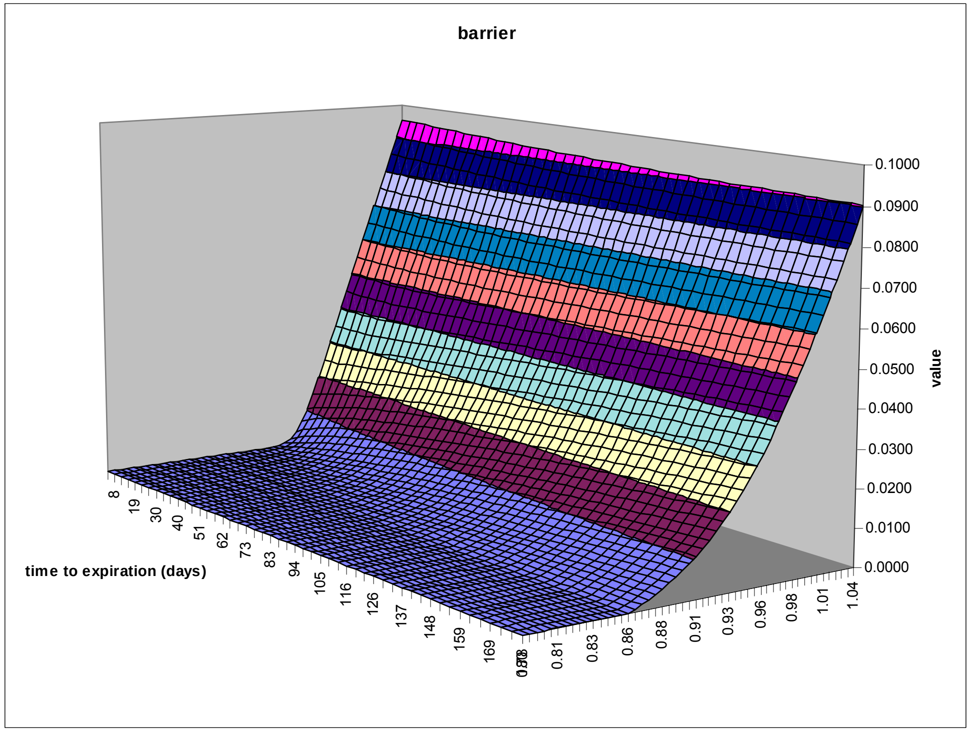
- Vanilla delta is between 0 and 100%
- So if we sell a EUR call USD put with delta 40% we need to
- buy 40% EUR of the notional

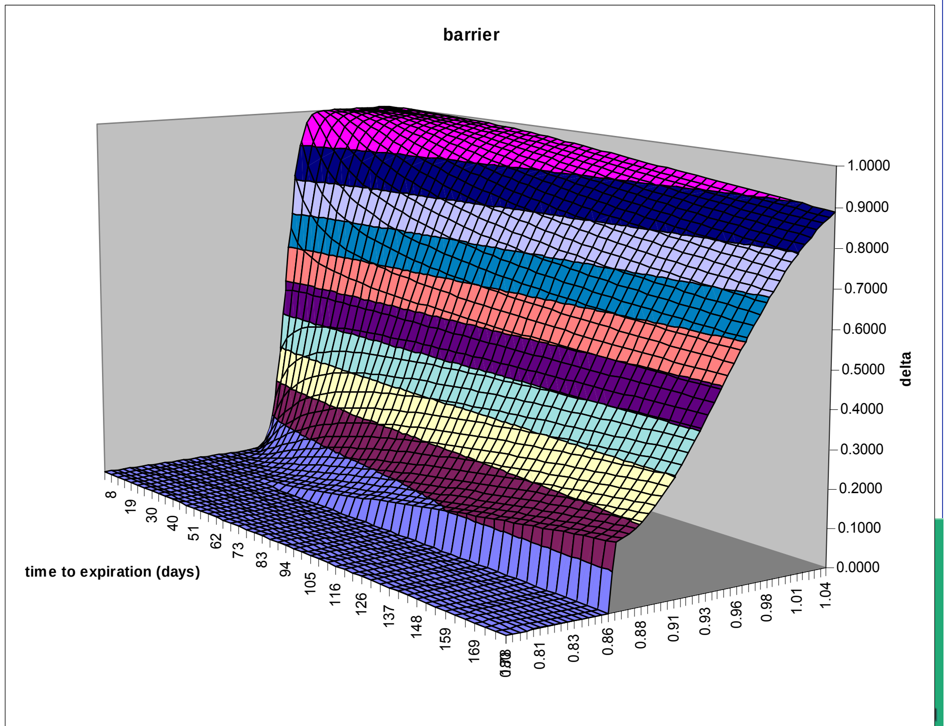




Hedging Barrier Options

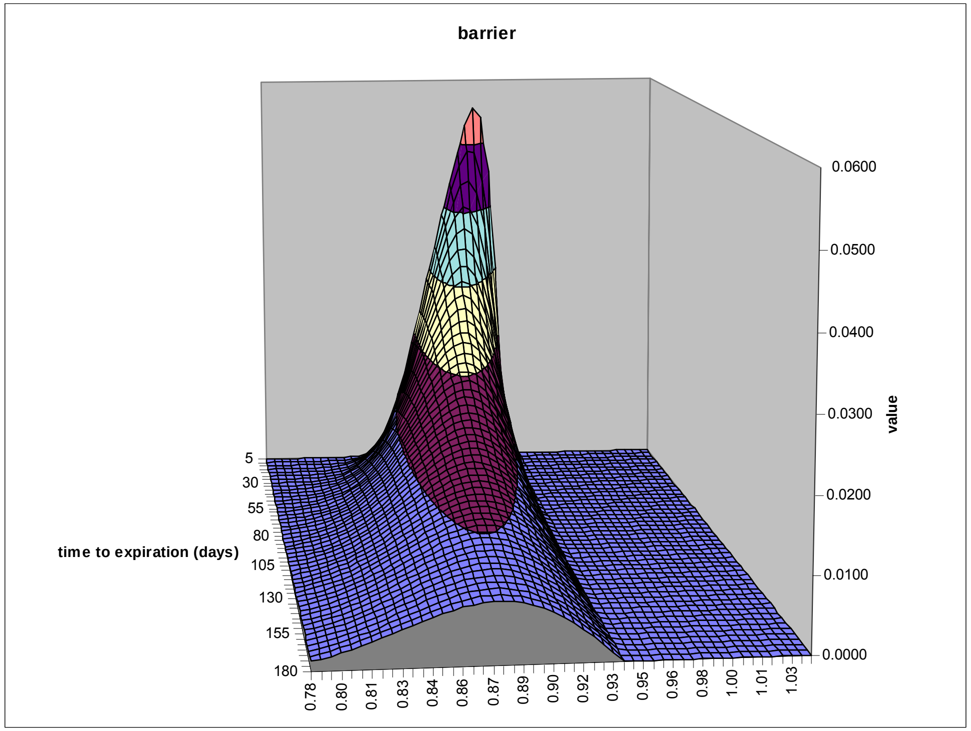
- Vanilla delta is between 0 and 100%
- How about barrier option deltas?
- No problem for regular barriers:

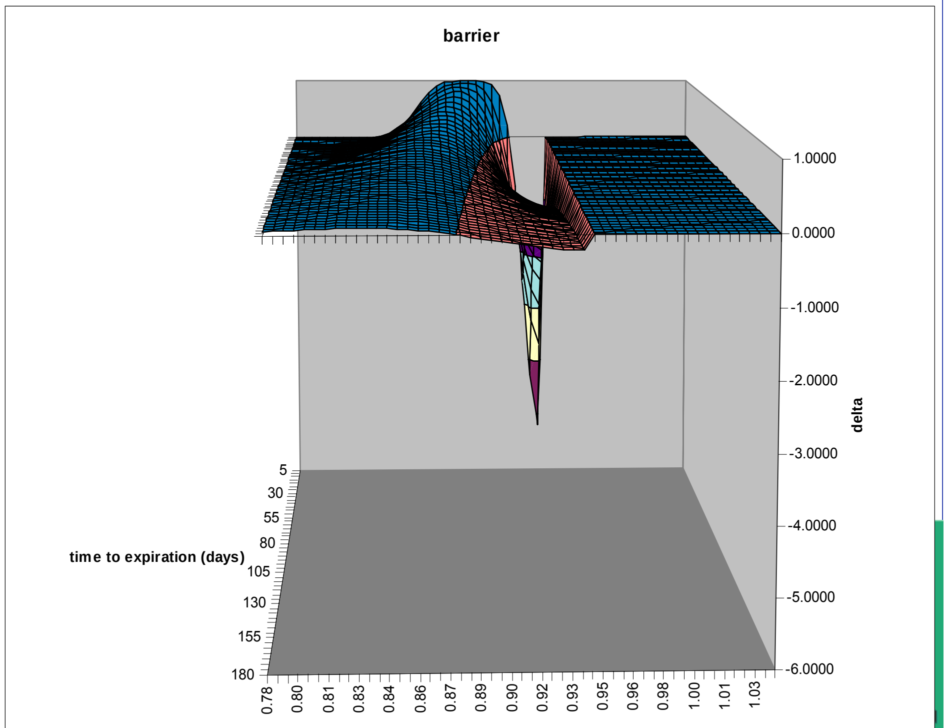




Hedging Barrier Options

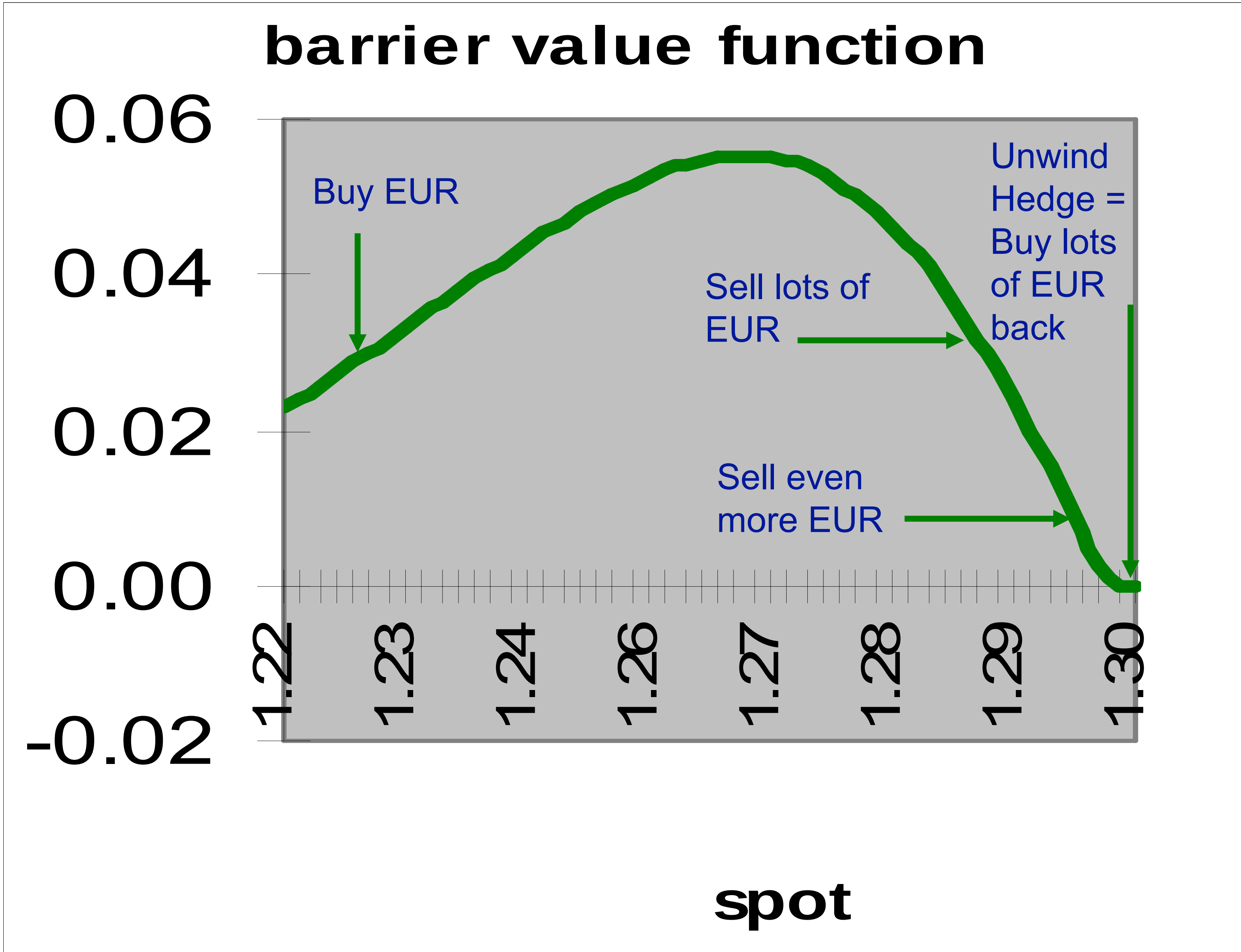
- Vanilla delta is between 0 and 100%
- How about reverse barrier option deltas?
- They can become arbitrarily large !





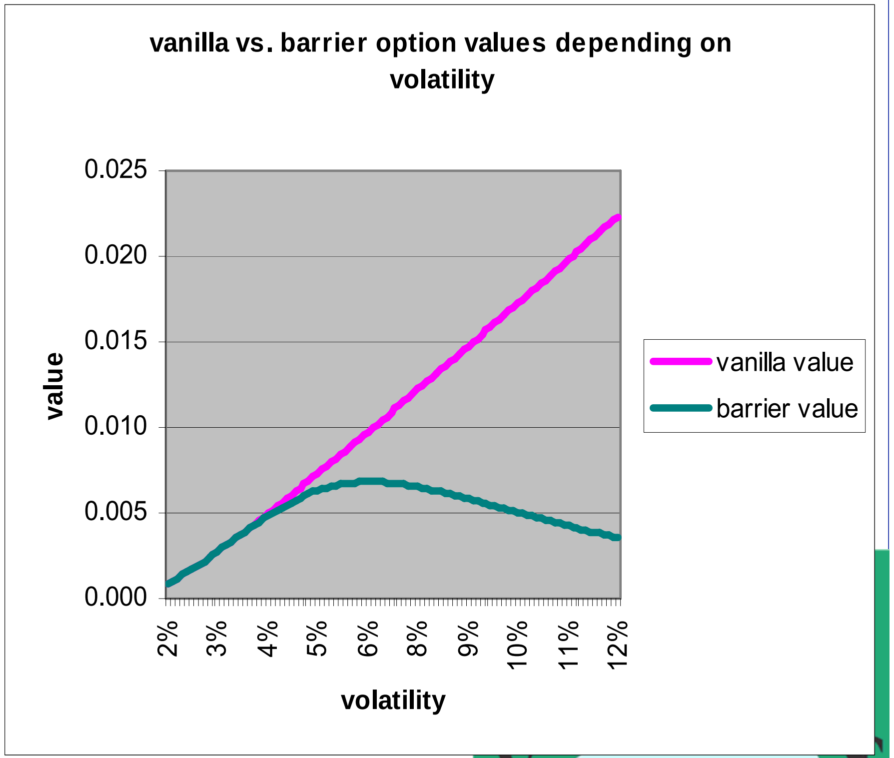
How large barrier contracts affect the market

- Example: reverse down-and-out put in EUR/USD with strike 1.3000 and barrier 1.2500.
- An investment bank delta-hedging a short position with nominal 10 Million has to buy 10 Million times delta EUR.
- As the spot goes down to the barrier, delta becomes larger and larger requiring the hedging institution to buy more and more EUR.
- This can influence the market since steadily asking for EUR slows down the spot movement towards the barrier and can in extreme cases prevent the spot from crossing the barrier.
- Once the barrier is breached, the bank has to unwind the delta hedge, sell lots of EUR -> rate goes down fast.


barrier value function
0.06
0.04
0.02
0.00
-0.02
Buy EUR
Sell lots of EUR
Sell even more EUR
Unwind Hedge = Buy lots of EUR back
1.22
1.23
1.24
1.26
1.27
1.28
1.29
1.30
spot

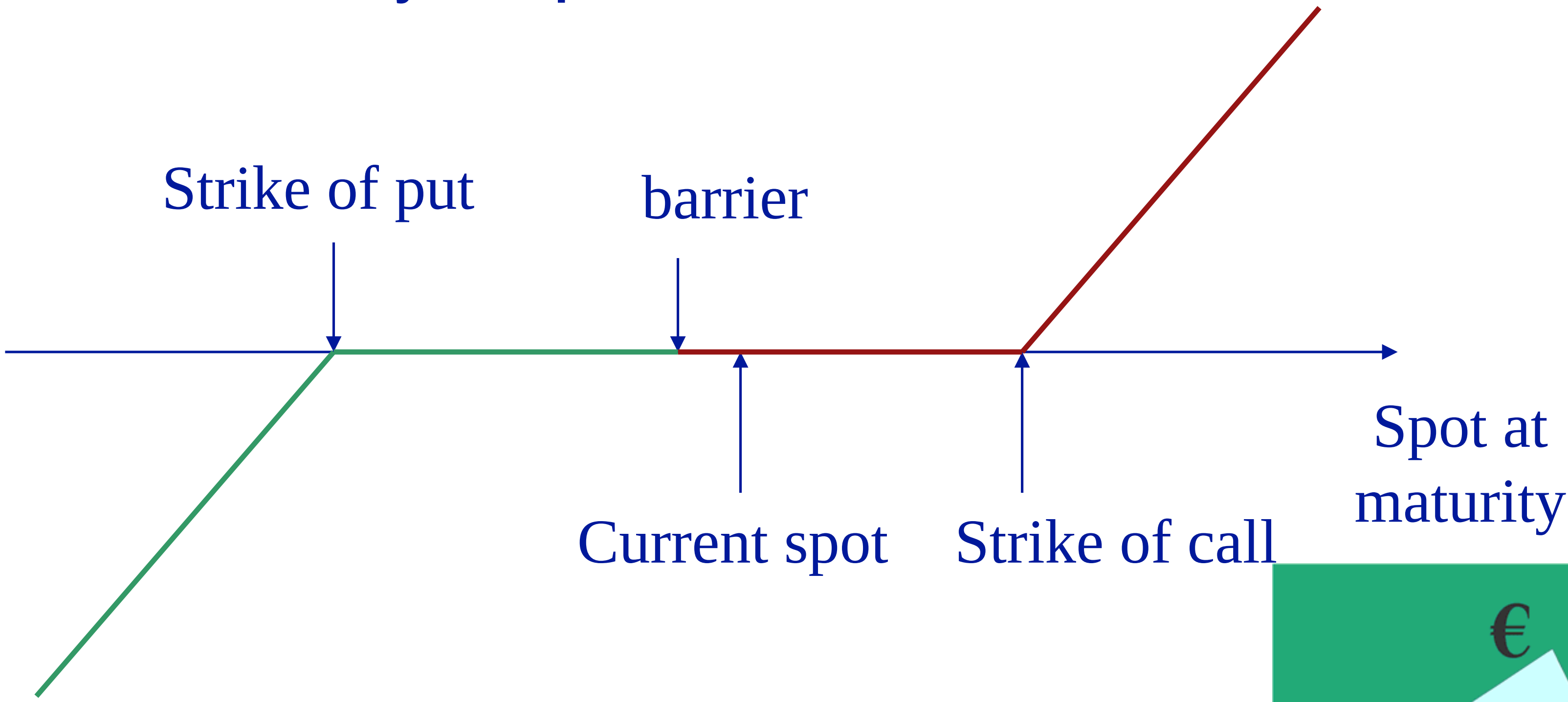
Hedging Barrier Options

- Which volatility should one take to price barrier options?
- Or: is there a smile for barrier options?
- Answer: not so easy as vol->price is not monotone!
- Thus: Given the price
- the volatility is not unique
- Open question:
- Pricing of barrier options
- Answer:
- look at hedge cost!



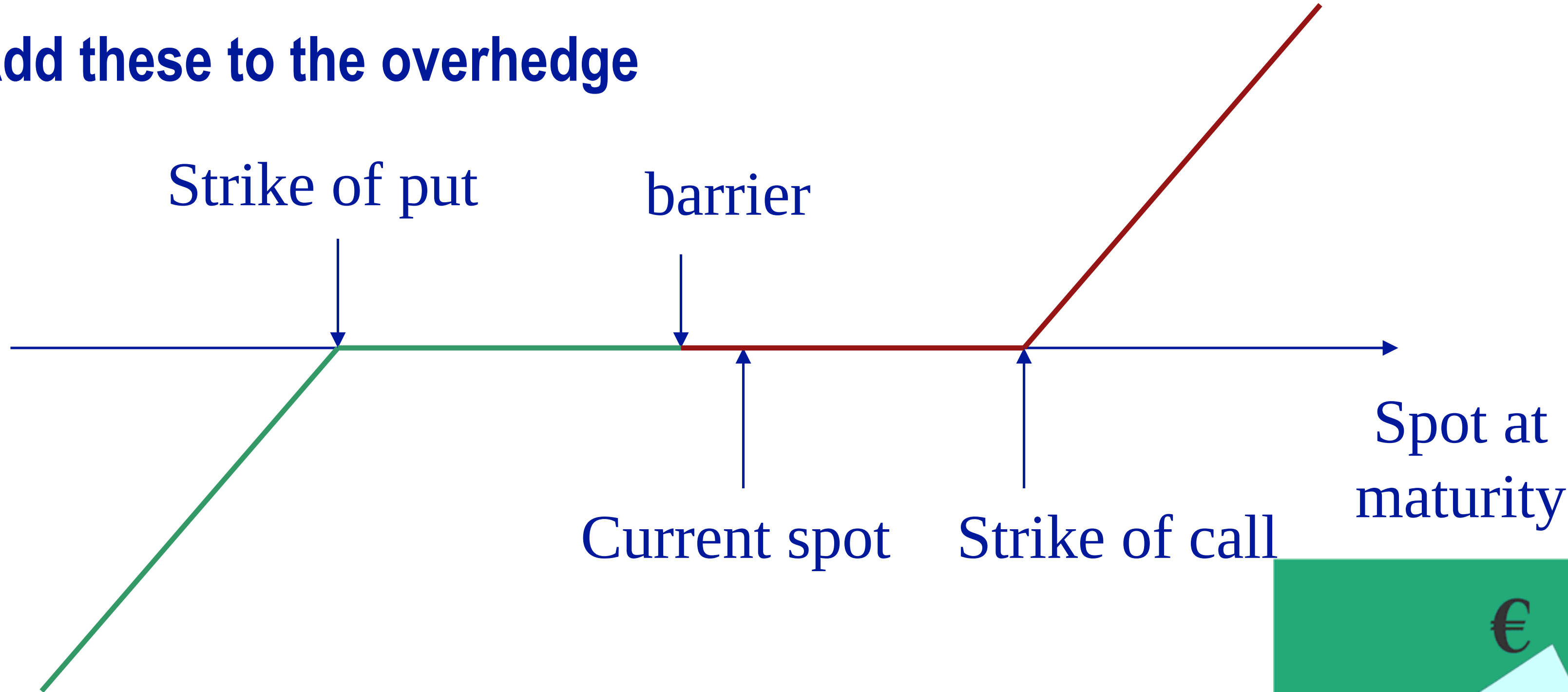
Hedging Barrier Options

- E.g. a regular knock-out call with a risk reversal (RR)
- Price of the RR is an indication for the price of the knock-out
- And vanilla smile yields price of the RR



Hedging Barrier Options

- No KO: Perfect static hedge
- KO: ideally: unwind cost of RR = 0. Unfortunately not realistic
- Determine expected unwind cost from experience (e.g. time series analysis / regression of log-returns)
- Add these to the overhedge



Pricing RKO Barrier Options

- **Compute theoretical value (TV)**
- **Adjust value based on hedge cost (volatility management)**
- **Add supplement for delta hedging difficulty**
- **The latter is approximately the same as: Instead of RKO with barrier B price an RKO with barrier B(1+a), a=1% or a=0.5% depending on risk appetite, details in** ***Dealing with Dangerous Digitals*** **in** ***Foreign Exchange Risk***
- **Discussion of static approaches**

Moving towards structuring: Exercise 1

Replicate a double-no-touch using a portfolio of double-knock-out options.

The nominal amounts of the respective double-knock-out options depend on the currency in which the payoff is settled.

In case of a EUR-USD double-no-touch paying 1 USD **(**domestic currency**)**, determine the nominal amounts of the required double-knock-out calls and puts

How do you replicate a double-no-touch paying one unit of EUR **(**foreign currency**)**?

Moving towards structuring: Exercise 2

Replicate a digital call using vanillas.

How does it work?

What are the problems?

What does this imply for the market price of digitals?

Can we use the same smile volatility for digitals as for vanillas if the strike is the same?

Moving towards structuring: Exercise 3

Replicate European style barriers using digitals and vanillas.

Replicate a KIKO (knock-in-knock-out) with barrier options.

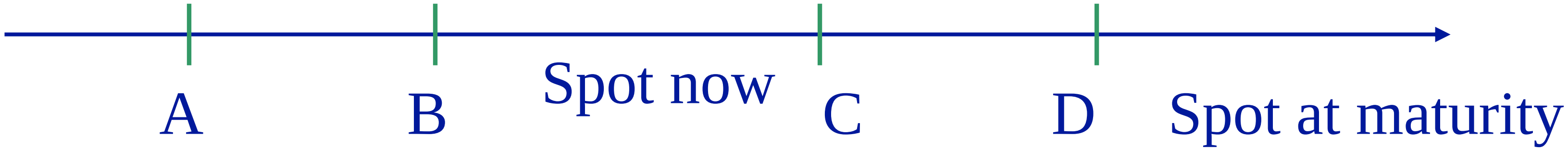
In a KIKO one barrier is a knock-out barrier, the other one a knock-in barrier. Any special concerns?

Replicate a Transatlantic barrier option with vanilla and barrier options. One barrier is of American, the other barrier of European style.

Moving towards structuring: Exercise 4

Replicate reverse knock-out barriers using a portfolio of OT, NT, DOT, DNT, KO, KI, Digitals, Vanillas. You are not allowed to use a RKI.

Problem: a tolerant Double No-Touch Option



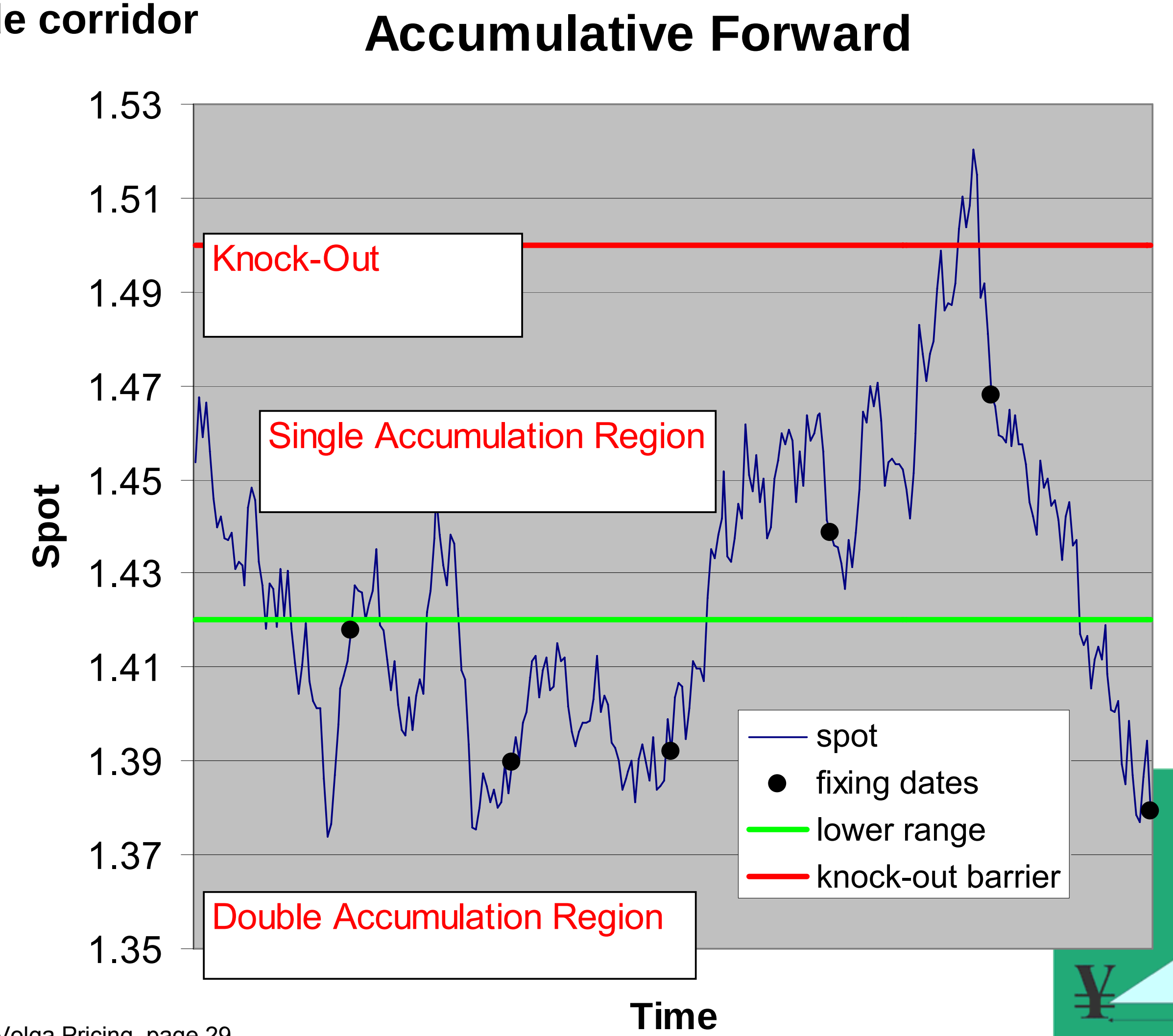
- **Given four barriers A, B, C, D**
- **tolerant double no-touch knocks out after the second barrier is touched or crossed**
- **How to replicate it using barrier and/or touch options?**

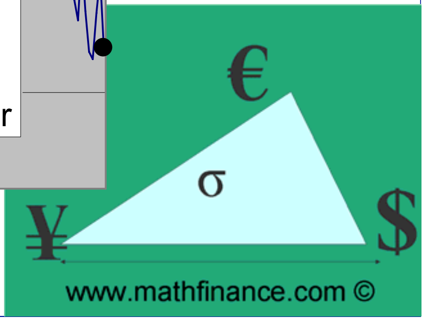
Accumulators

Consider Fixing Schedule $S_1, S_2, \ldots, S_N$

N = all Fixings

n = Fixings inside corridor





Accumulator: Pricing/Hedging

- **Start with Black-Scholes TV. Compose approximate hedge using first generation exotics, whose overhedge is known**
- **E.G. EUR/USD spot 0.9800 of September 24 2002. T=15 months. The client buys a total of 28 million EUR at an improved rate of 0.9150**
- **For each EUR/USD fixing between 0.9150 and 1.0500 the client accumulates 28 million EUR divided by the number of fixing days.**
- **For each EUR/USD fixing below 0.9150 the client accumulates twice this daily amount, such that in the extreme case of all fixing below 0.9150 the total amount accumulated would be 56 million EUR.**
- **If the non-resurrecting knock out level of 1.0500 is ever traded, then the accumulation stops, but the client keeps 100% of the accumulated amount.**
- **TV: client receives 400,000 EUR**

Accumulator: Pricing/Hedging – Overhedge Computation

- **For the 0.9150 EUR calls RKO at 1.0500 we determine the overhedge as the average of the maturities 6 to 15 months**
- **We do the same for the 0.9150 EUR puts knock out at 1.0500**

Tenor	Basis Points (in EUR) RKO calls	Basis Points (in EUR) KO puts
6 months	+25	-5
9 months	+40	-10
12 months	+40	-20
15 months	+40	-20
average	+36	-12

- **On 28 million EUR 36 basis points (bps) are 101,000 EUR, which is the overhedge for buying the RKO calls in the hedge**
- **Similarly on 56 million EUR 12 bps are 68,000 EUR, which is the overhedge for selling the KO puts. Both are priced at mid market, so fairly aggressive.**

Accumulator: Pricing/Hedging – Overhedge Computation

- cost of knock out: If the knock out level of 1.0500 is reached, the client has the right to buy the accumulated EUR amount at 0.9150 (with the pre-determined value date), even though the spot is then at 1.0500.

- For the bank selling the accumulative forward this is substantial risk, which can be hedged by buying a 1.0500 one-touch with 15 months maturity.

- The only thing is the notional of this one-touch has to be approximated. First of all the amount at risk, called the parity risk is

  1.0500 - 0.9150 = 0.1350 USD per EUR = 12.86 % EUR.

- We approximate the time it takes to reach the parity level of 0.9150 by 7 months and take from the market that the price of a 7 month 0.9150 one-touch is 40% .

Accumulator: Pricing/Hedging – Overhedge Computation

- So say 40% chance below 0.9150,
accumulating 22.23 million EUR

- 60% chance above 0.9150, accumulating 16.8 million EUR
- The sum of these two amounts may be the total
of 39.03 million EUR accumulated.
- The 15 months 1.0500 one-touch would cost 53.5%
- parity risk amount equals 39.03 million * 53.5% * 12.86% = 2.7 million EUR
- The one-touch overhedge would be 2.7 million * 3% (mid market) = 81,000 EUR
- to hedge the vega of 206,000 EUR,
we take the bid-offer spread of the price in volatilities and
arrive at 206,000 * 0.15 vols (bid-offer) = 31,000 EUR.
- →total overhedge = 101,000 + 68,000 + 81,000 + 31,000 = 281,000 EUR

Stochastic Volatility

Why stochastic volatility?

Because volatility is stochastic!

E.g.: USD/JPY 1M ATM impl. Vol 1994-2000

Hull/White (1987)

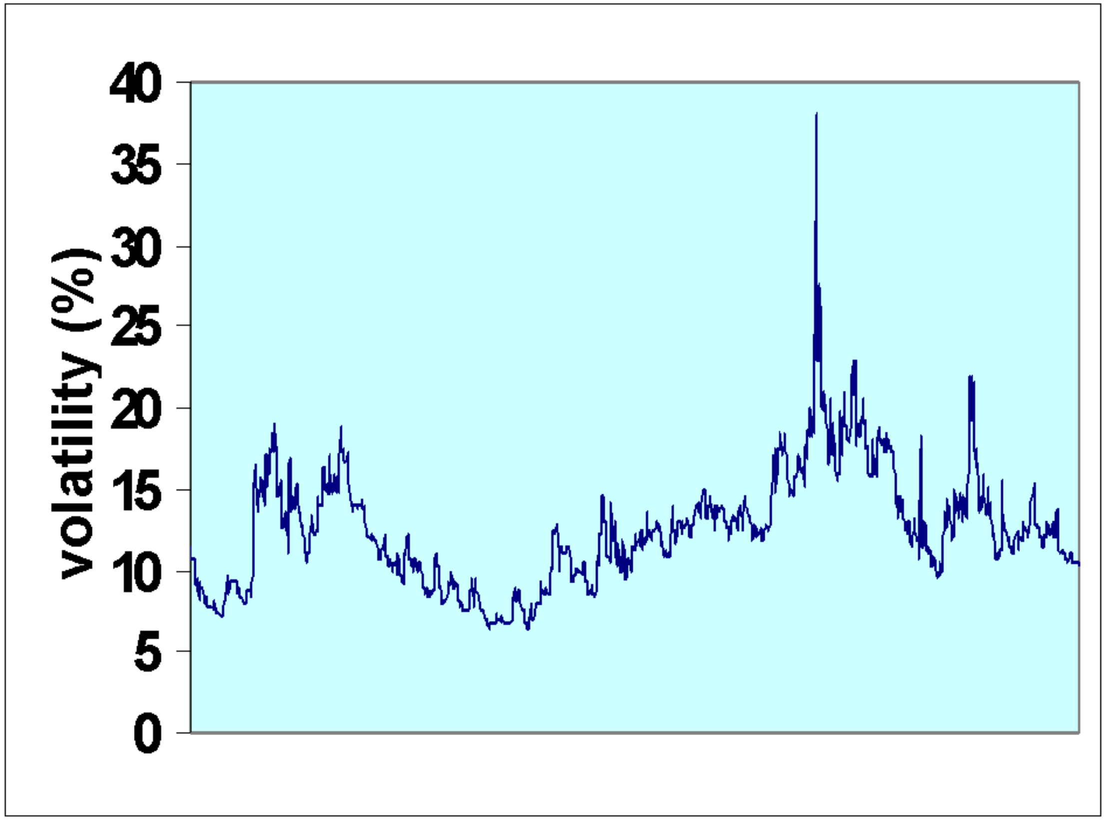
Stein/Stein (1991)

Heston (1993)

Schöbel/Zhu (1998)

Hagan (2000)

...



Heston's Model

- σ Instantaneous volatility
- κ Mean reversion speed
- θ Long-term instantaneous variance
- ζ Volatility of variance (vol of vol)
- ρ Correlation

$$dS_t = (r_d - r_f) S_t\,dt + \sigma S_t\,dW_t^1$$

$$\sigma = \sqrt{V}$$

$$dV_t = \kappa(\theta - V_t)\,dt + \zeta \sqrt{V_t}\,dW_t^2$$

$$dW_t^1 dW_t^2 = \rho\,dt$$

Heston's Model

Call value $H(t, S, \sigma)$

Satisfies the PDE

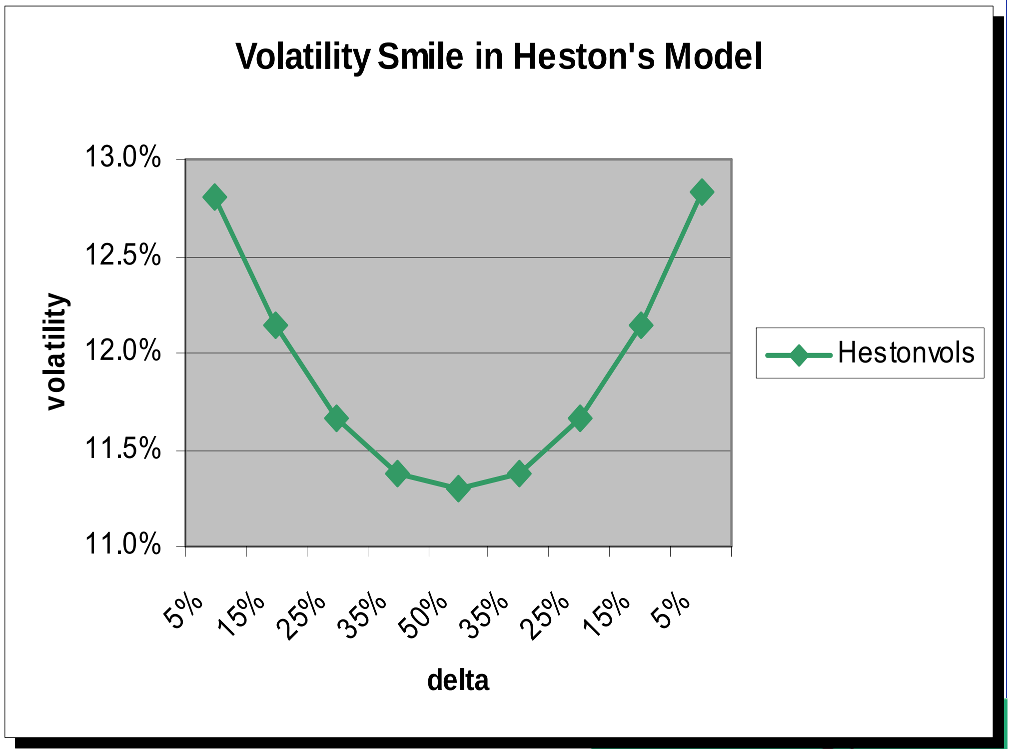
$$H_t + (r_d - r_f) S H_S + \frac{1}{2} S^2 \sigma^2 H_{SS} - r_d H$$

$$+ \frac{1}{2} \zeta^2 V H_{VV} + \rho S \zeta V H_{VS} + [\kappa(\theta - V) - \lambda V] H_V = 0$$

λ **Market price of volatility risk, can be set to zero in the calibration**

Advantages of Heston‘s Model

- Can be implemented
- Covers wide product range
- Explains market prices

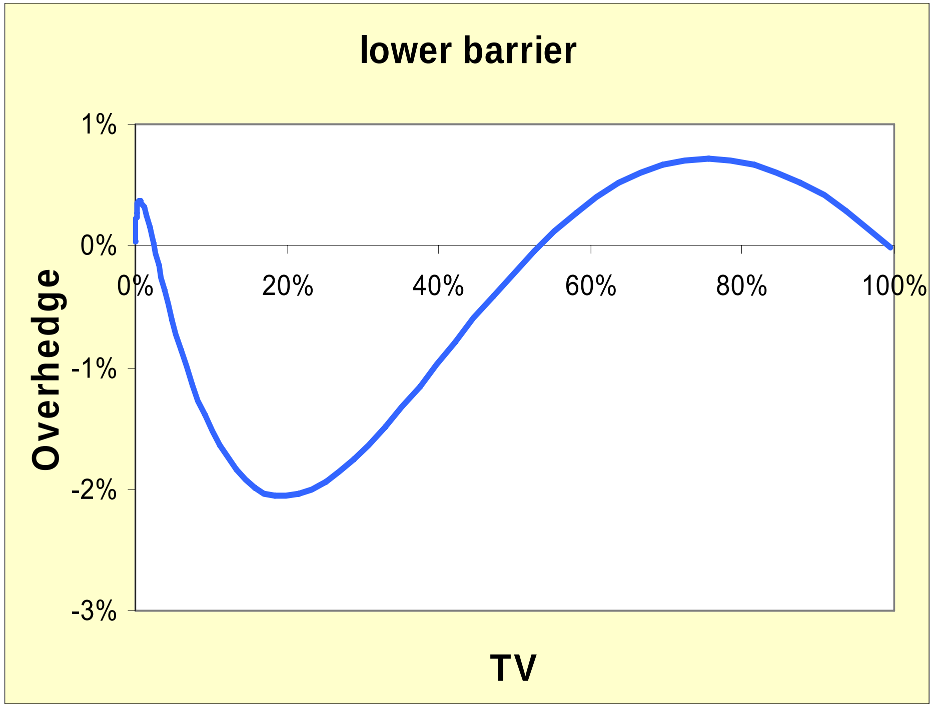


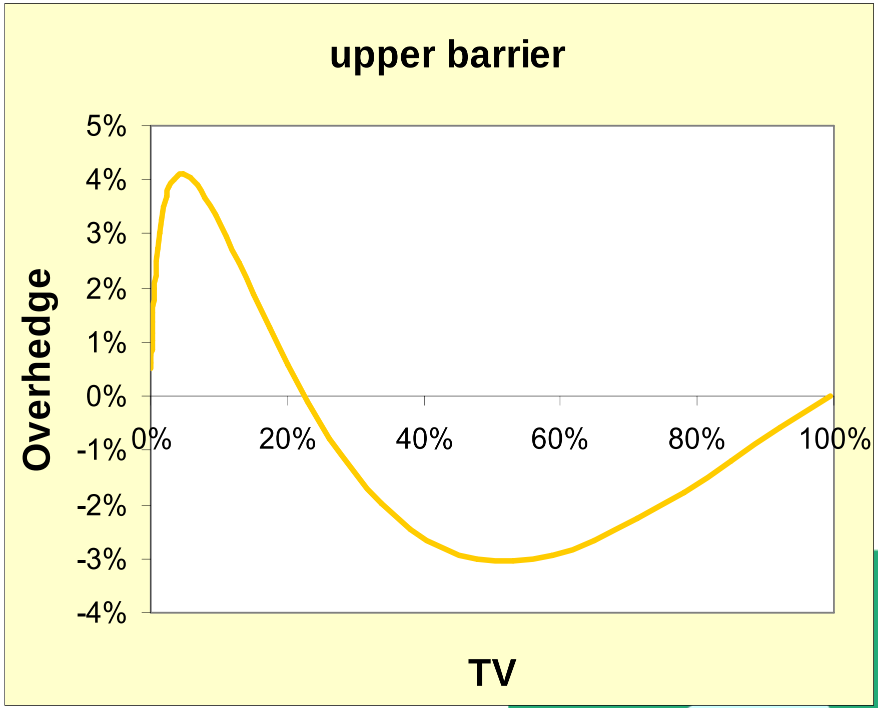
Case Study: One-touch with Heston vs Vanna-Volga

- Pays a fixed amount of a pre-specified currency (EUR or USD), if EUR/USD touches or crosses a barrier at any time up to expiration
- Price is between 0% and 100% of the notional
- The closer the spot to the barrier, the higher the price of the one-touch
- Notional is paid at maturity (standard) or at first hitting time
- Price is often far from theoretical value, why?
- Cost of risk managing the volatility exposure

One-touch

- Price is often far from theoretical value (**TV**) , why?
- Cost of risk managing the volatility exposure (**Overhedge**)
- **Examples with lower and upper barrier**





Market data: EUR/USD 17 July 2002 1.0045 EUR 3.33% USD 1.76%, 3 M ATM vol 11.85%, RR 1.25%, BF 0.25%

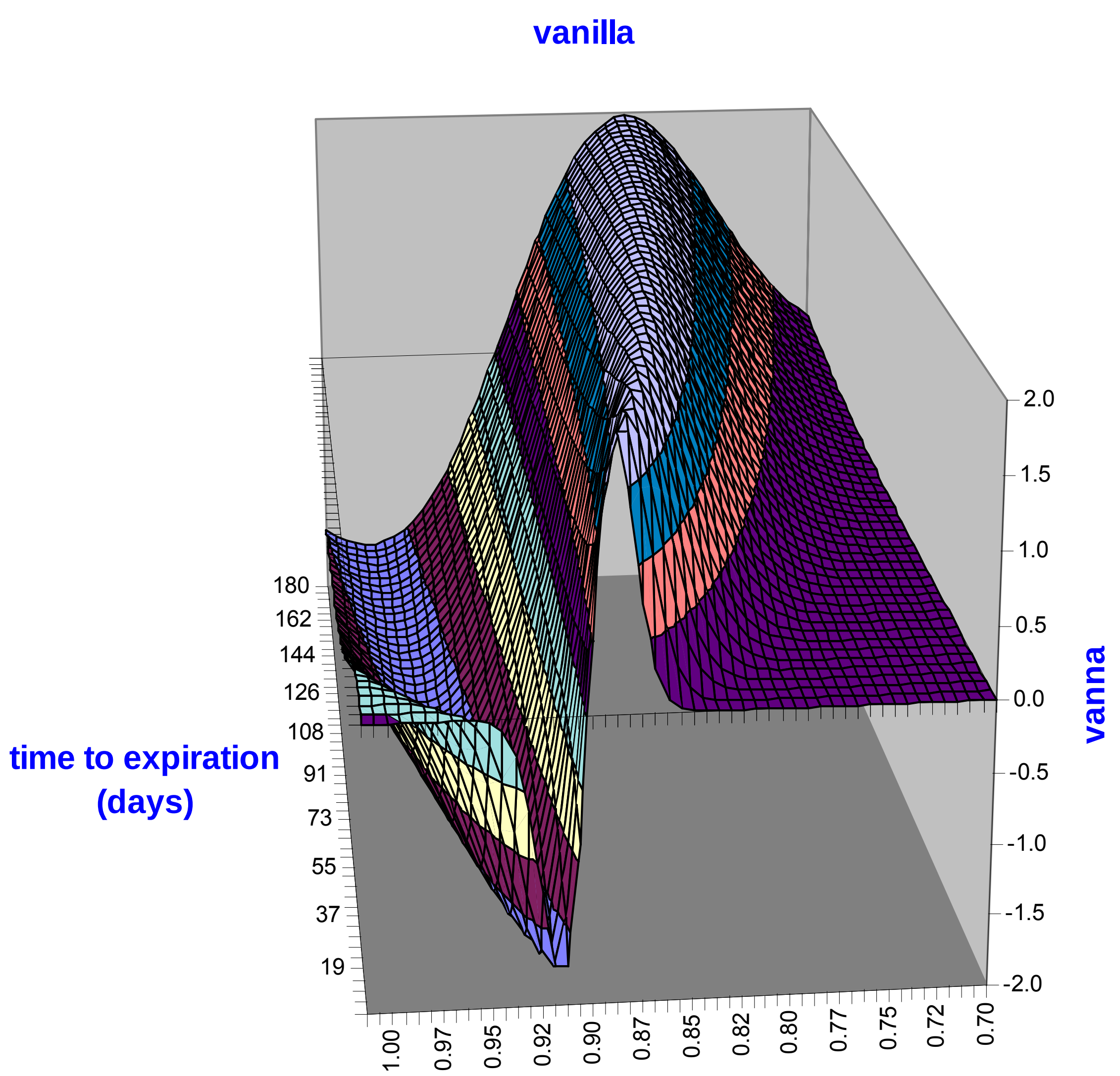
How does vanna-volga work ?

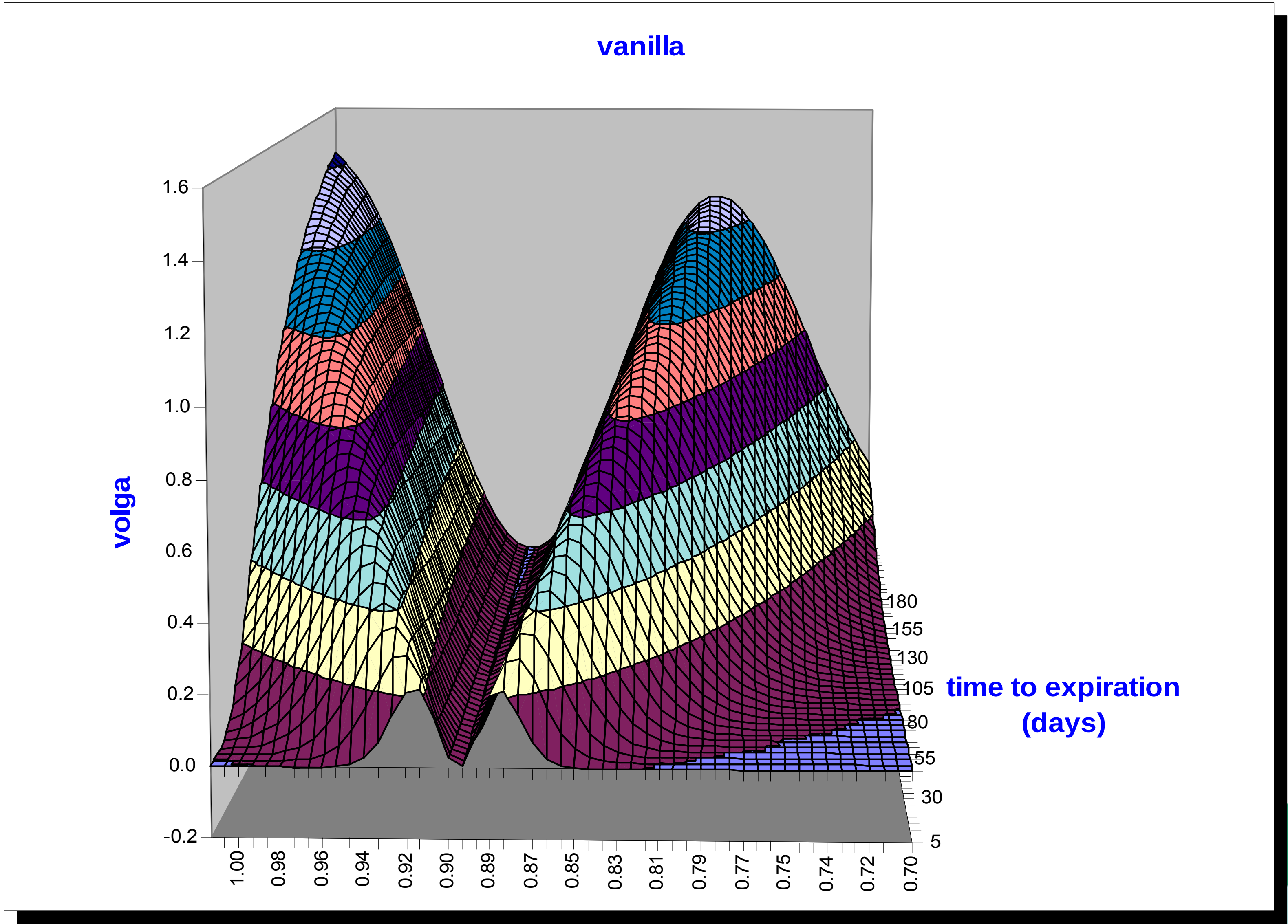
- **Market price of an exotic Option**
- **= TV**
- **+ p • vanna of the Option • OH RR / vanna RR**
- **+ p • volga of the Option • OH BF / volga BF**

- **RR: Risk Reversal**
- **BF: Butterfly**
- **p: Probability that the hedge is needed**
- **OH: overhedge = market price – Black-Scholes TV**

Volatility Risk for Options

- Goal: Compute the cost of vega management
- Vanna = change of vega when spot changes
- → compute the cost of vanna
- Volga = change of vega when volatility changes
- → compute the cost of volga
- Overhedge to Black-Scholes TV = total cost of vanna and volga


vanilla
time to expiration (days)
vanna
180
162
144
126
108
91
73
55
37
19
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
1.00
0.97
0.95
0.92
0.90
0.87
0.85
0.82
0.80
0.77
0.75
0.72
0.70


vanilla
volga
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
1.00
0.98
0.96
0.94
0.92
0.90
0.89
0.87
0.85
0.83
0.81
0.79
0.77
0.75
0.74
0.72
0.70
180
155
130
105
80
55
30
5
time to expiration (days)

Vanna-Volga Literature

- Lipton, A. and McGhee, W. (2002).
  Universal Barriers. Risk, May 2002.
- Wystup, U. (2003).
  The Market Price of One-touch Options in Foreign Exchange Markets.
  Derivatives Week Vol. XII, no. 13, London.
- Castagna, A. and Mercurio, F. (2007).
  The Vanna-Volga Method for Implied Volatilities.
  Risk, Jan 2007, pp. 106-111.
- Patent file of SuperDerivatives (google!)

How does vanna-volga work ?

- Example: USD/JPY 1-year one-touch at barrier 127.00 with Nominal in USD
- Market data: spot 117.00, Volatility 8.80%, USD rate 2.10%, JPY rate 0.10%, 25Delta RR -0.45%, 25 Delta BF 0.37%
- TV: 38.2%
- Vanna: -9.0
- Volga: -1.0

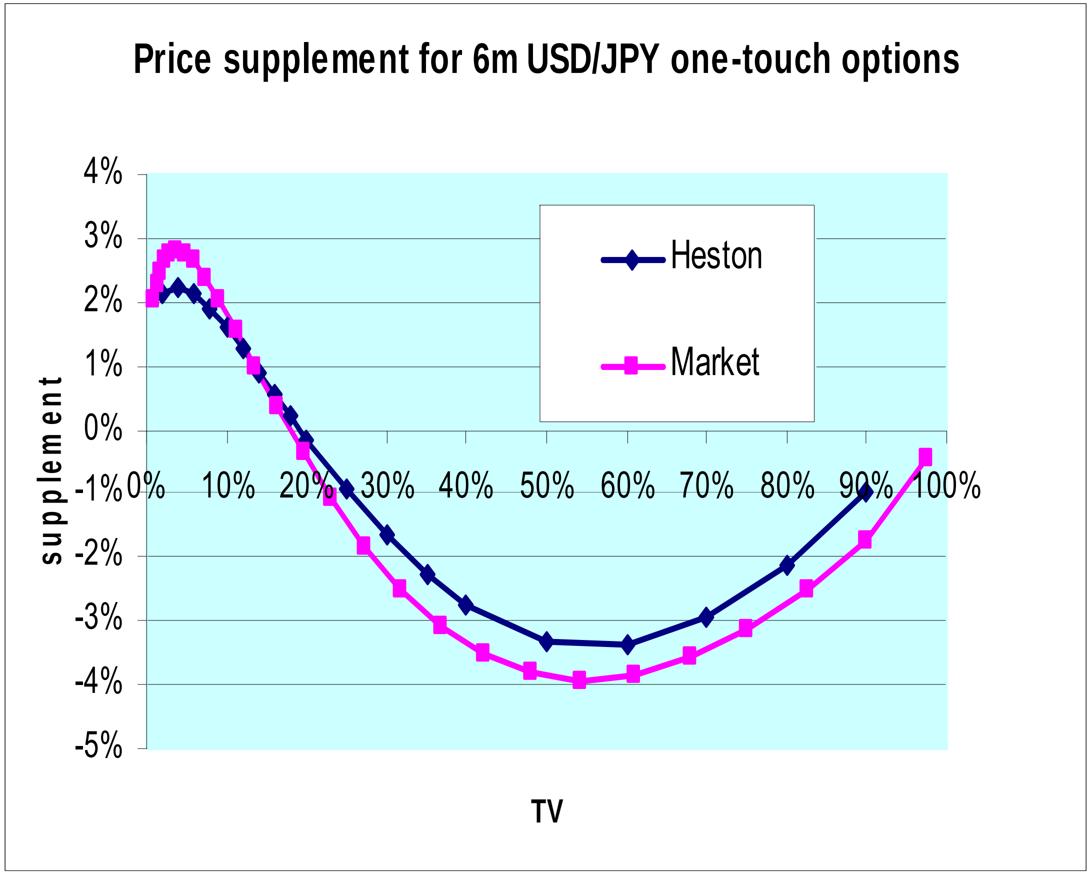
How does market-oriented valuation work ?

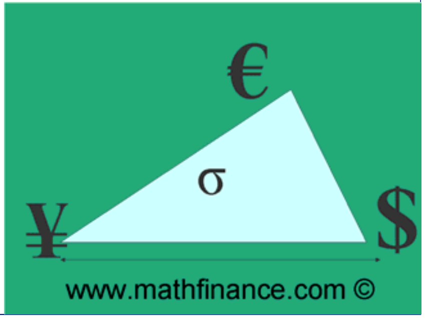
Market price of an exotic Option
- = TV
- \+ p • vanna of the Option • OH RR / vanna RR
- \+ p • volga of the Option • OH BF / volga BF

- **Example USD/JPY one-touch**
- = 38.2%
- \+ p • [-9.0 • (-0.15%) / 4.5]
- \+ p • [-1.0 • 0.27% / 0.035]
- = 38.2% + p • [0.3% - 7.7%]
- = 38.2% - p • 7.4%

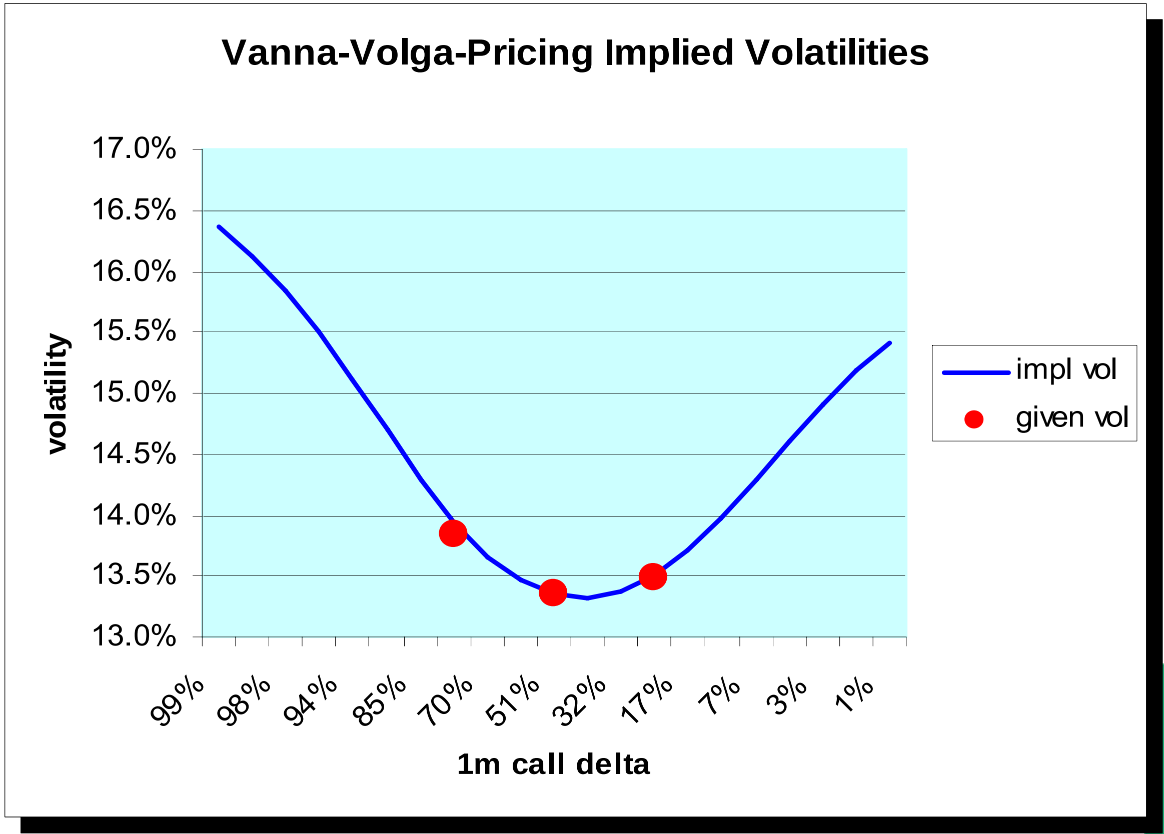
- **Hitting probability: 38.2%**
  - Hedge is not needed with 38.2% probability
  - Thus, p = 100% - 38.2% = 61.8%
  - Total overhedge: 61.8% • -7.4% = -4.6%
  - Market price: 38.2% - 4.6% = 33.6%
  - Bid/Ask: 32% / 35%
- **Hedge of a long position: sell 2 RR and 28 BF**

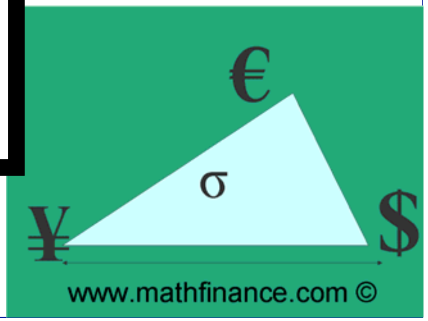
Heston vs. Market for the One-Touch



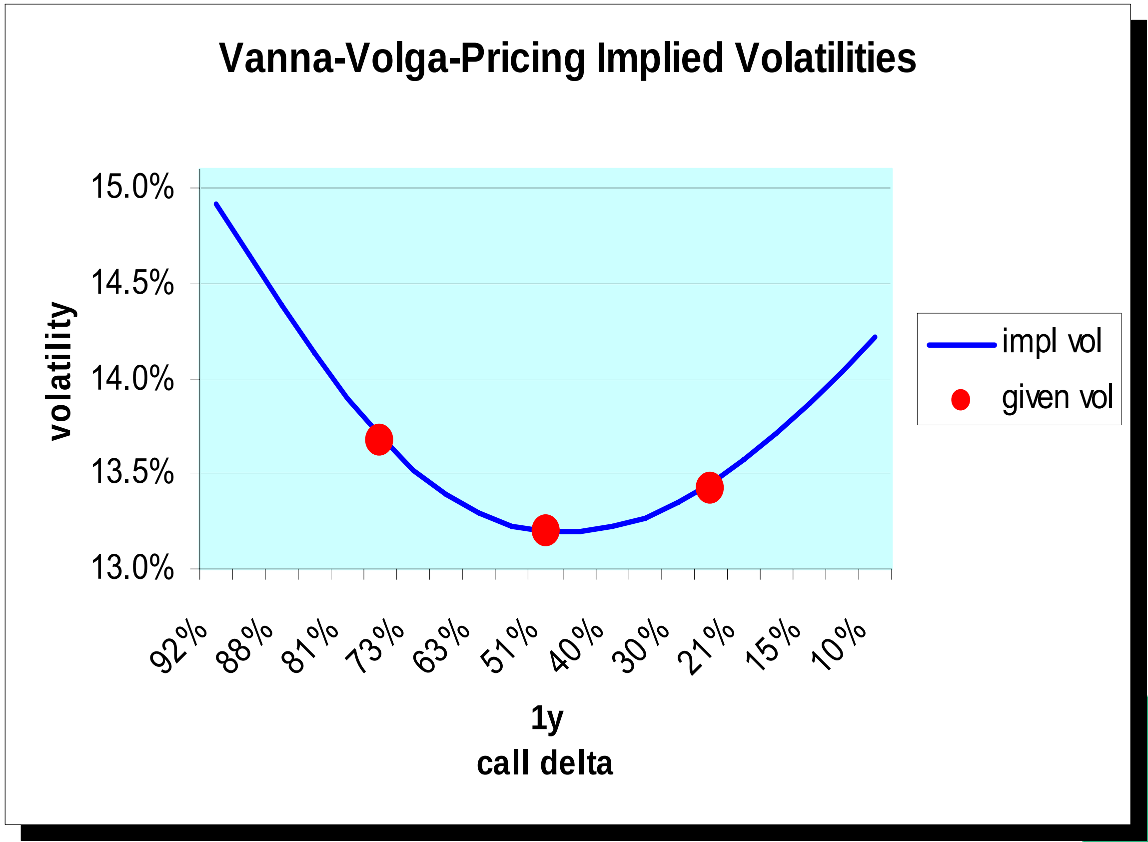


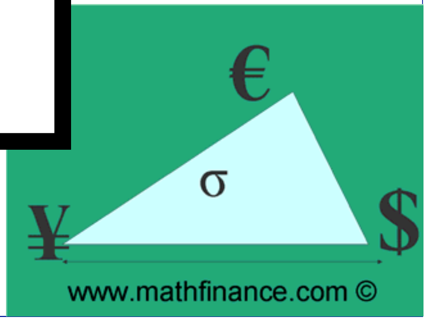
Is Vanna Volga Pricing consistent with the Smile?



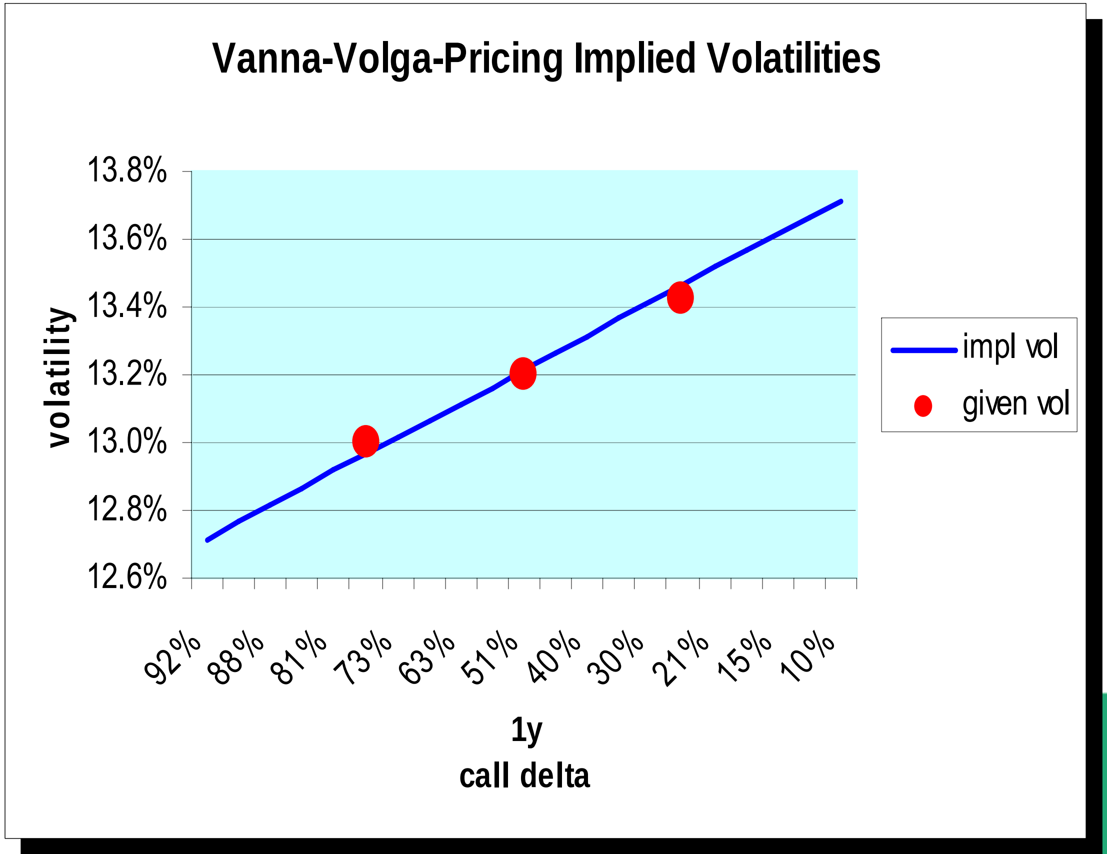


Is Vanna Volga Pricing consistent with the Smile?





Is Vanna Volga Pricing consistent with the Smile?



What to take for p?

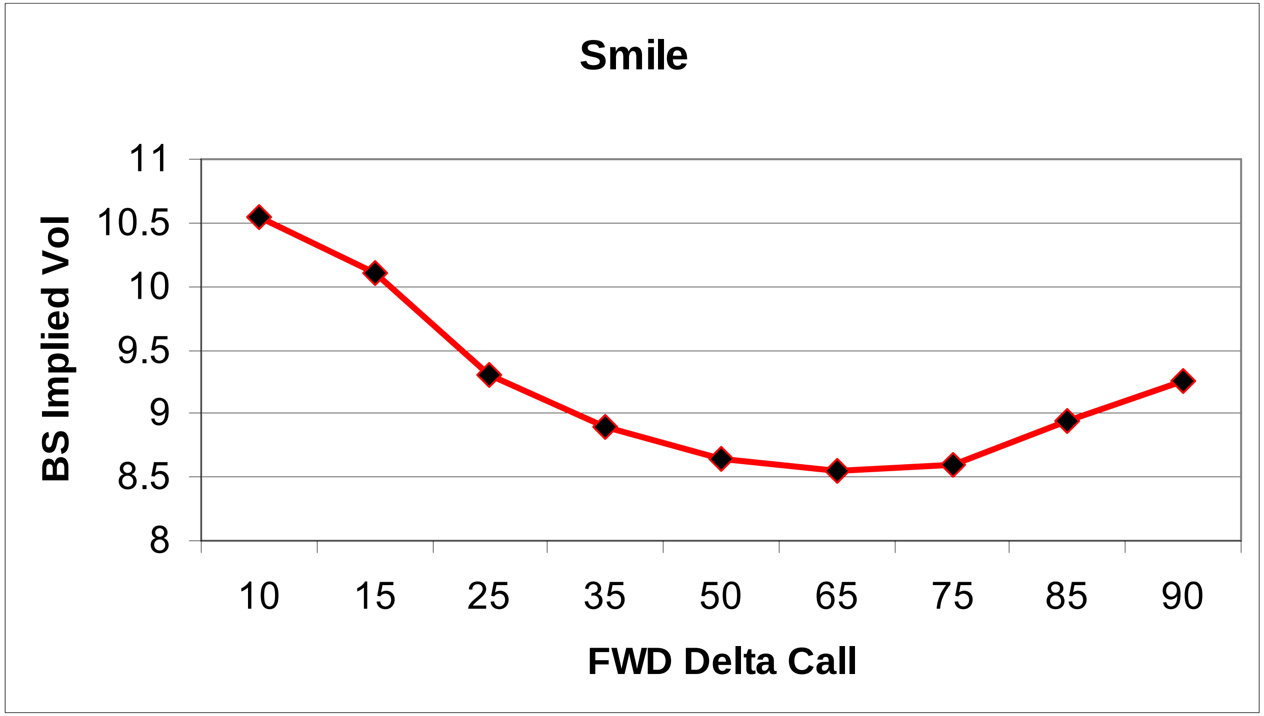
Product	p
KO, RKO, DKO	No-Touch probability
DNT	0,5
OT	0,9*No-Touch- probability – 0,5*bid-ask-spread*(TV-33%)/66%

First Generation Exotics

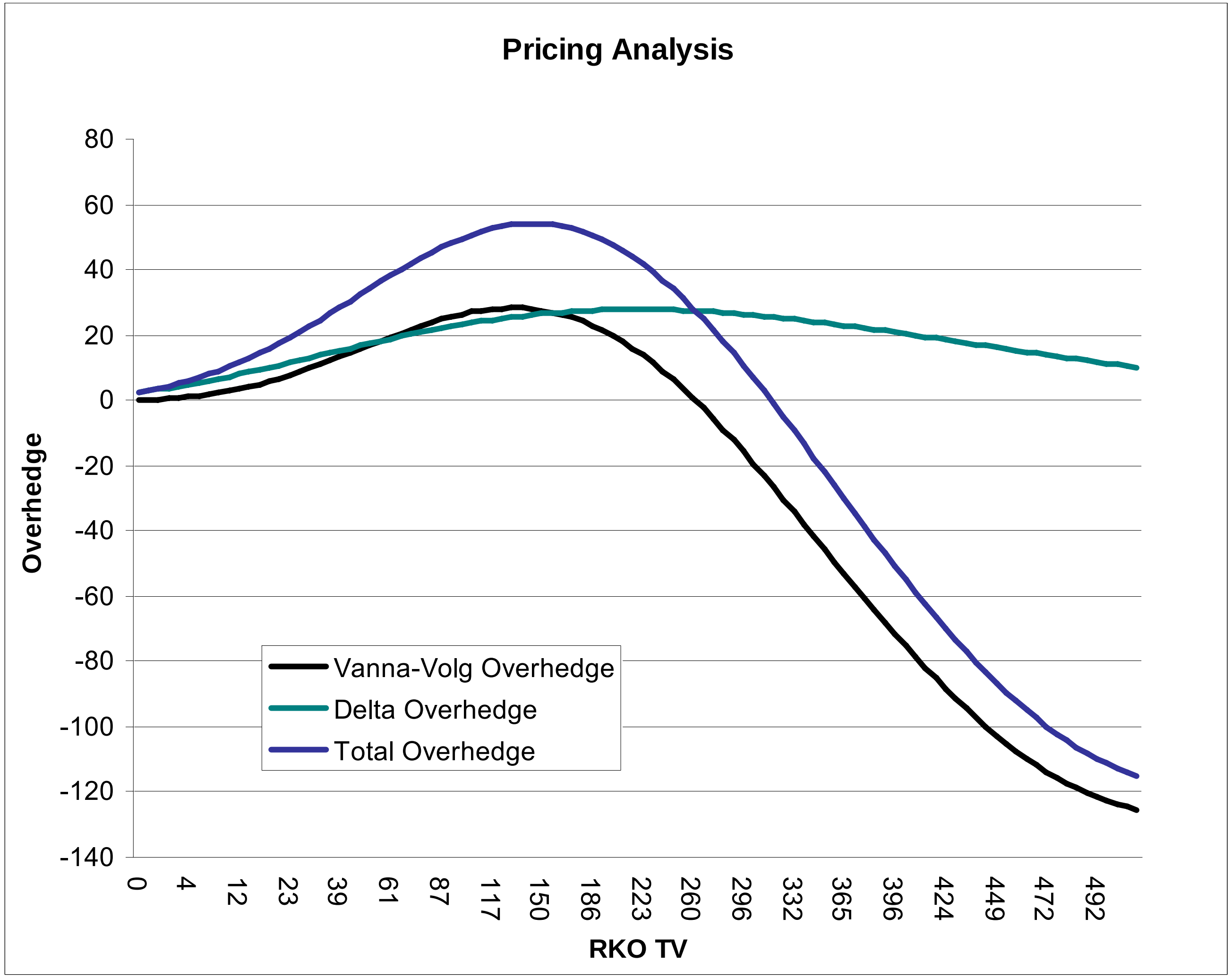
Product	Valuation via
KI, RKI	Vanilla – KO, Vanilla - RKO
RKO	KO, digital options
DOT	1 - DNT
NT	1 - OT
European barrier options	Vanilla and digital options
Digital options	Vanilla Spread

Example: Reverse Knock Out EUR call USD put

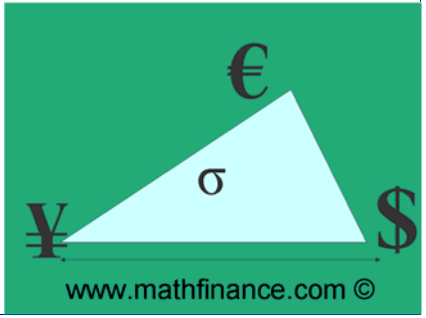
- T=182 days
- Strike 1.5000
- AMT=10kEUR
- Spot 1.5500
- Feb 28 2008
- USD=2.99%
- EUR=4.43%
- ATMFWD=8.64%



Example: Reverse Knock Out EUR call USD put



- E.g. Barrier 1.6500, TV 215, OH(vv) 16
- Mid Market 231, OH(delta) 28



Basket Call: Case Study

- **Scenario: USD, GBP and JPY to be changed into EUR**
- **Market volatilities (from Reuters) are**

Market data	2-Jul-04
volatilities	FX pair
9.00	GBP/USD
9.90	USD/JPY
10.40	GBP/JPY
10.10	EUR/USD
7.40	EUR/GBP
10.30	EUR/JPY

Basket Call: Case Study

- **Scenario: USD, GBP and JPY to be changed into EUR**
- **Maturity: 3 months**
- **FX volatilities imply the correlations**

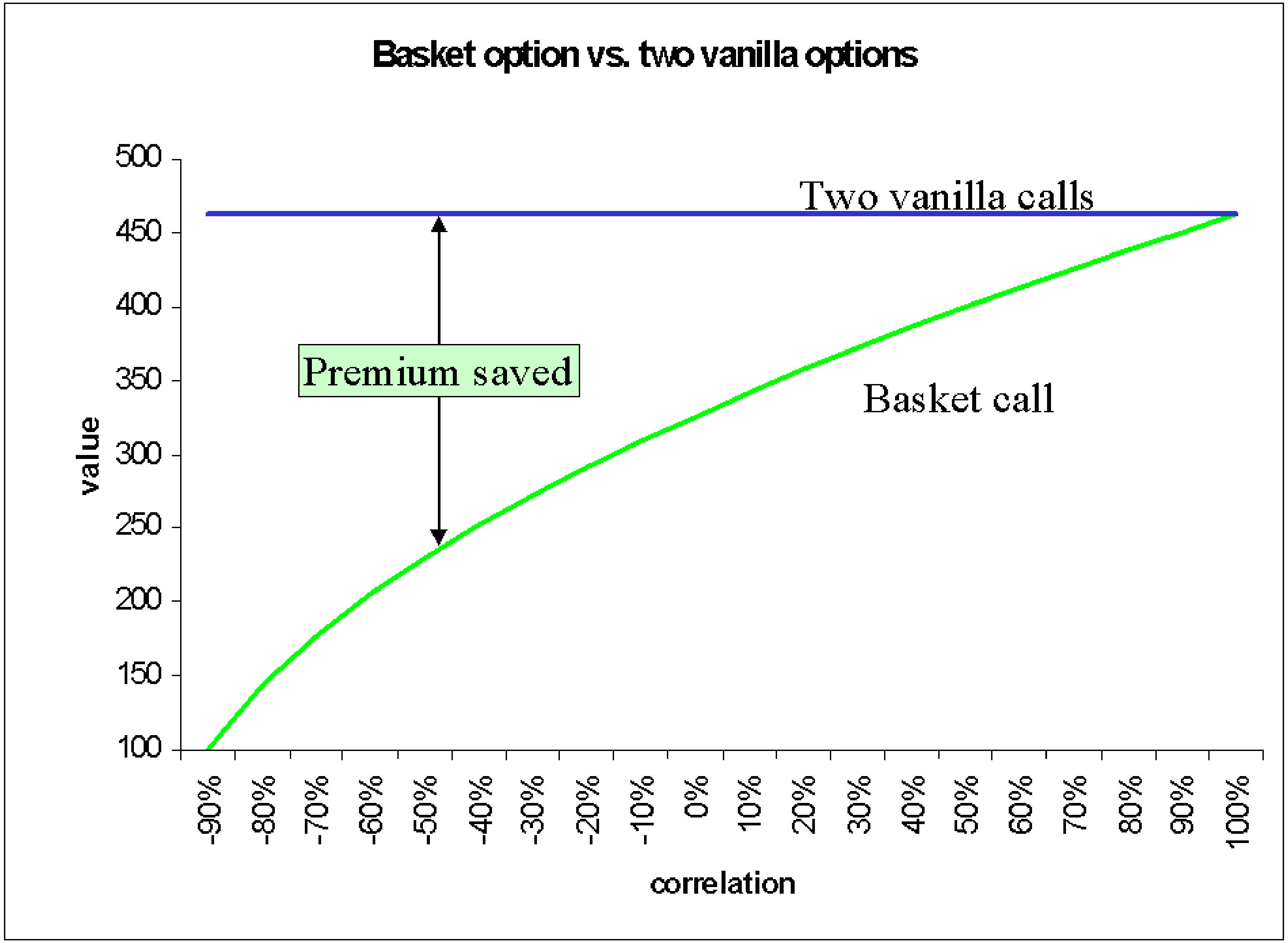
correlation		spot	1.2150	0.6690	132.50
GBP/USD	USD/JPY	GBP/JPY	EUR/USD	EUR/GBP	EUR/JPY
1.00	-0.40	0.49	0.71	-0.25	0.31
-0.40	1.00	0.61	-0.47	-0.16	0.50
0.49	0.61	1.00	0.16	-0.37	0.74
0.71	-0.47	0.16	1.00	0.51	0.53
-0.25	-0.16	-0.37	0.51	1.00	0.35
0.31	0.50	0.74	0.53	0.35	1.00

Basket Call: Case Study

- **Scenario: USD, GBP and JPY to be changed into EUR**
- **Comparing a basket put with 3 single vanilla puts**

Base Currency	EUR	Basket Put EUR call		rate	2.12%
notional	10,000,000.00				
currencies			USD	JPY	GBP
weights			33.33%	33.33%	33.33%
Spot			1.2150	132.50	0.6690
1/Spot			0.8230	0.0075	1.4948
Strikes in EUR			1.2195	133.33	0.6667
volatilities in %			10.10	10.30	7.40
interest rates in %			1.61	-0.04	4.86
		sum			
Vanilla Prices	EUR	178,000.00	60,000.00	50,000.00	68,000.00
Basket Price	EUR	140,000.00			
Save	EUR	38,000.00			

Basket Call: Premium Saved



Basket Options

- **TV: use approximations as in „Making the Best out of Multi Currency Exposure: Protection with Basket Options“ in *The Euromoney Foreign Exchange and Treasury Management Handbook 2002*, J. Hakala and U. Wystup.**

- **Market price (quick): quote 10 basis points wider**

- **Market price (more precise): compare with a portfolio of vanillas with strikes chosen such that the basket** *is approximated as good as possible*. **From these vanillas derive a) the overhedge and b) a hedge. Details on** *Formula Catalogue* **of http://www.mathfinance.com under basket**

Model for a Multi-Currency Market

- **N-dimensional geometric Brownian motion**
- **(1) GBP/USD**
- **(2) USD/JPY**
- **(3) GBP/JPY**
- **W : standard Brownian motion**
- μ: **Drift (from rates)**
- σ: **Volatilities**
- ρ: **Correlations**

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sigma_i S_t^{(i)} dW_t^i$$

$$dW_t^i dW_t^j = \rho_{ij} dt$$

$$i = 1, \ldots, N$$

For Option Valuation we need

- Interest rates ← Money market
- Volatilities ← Vanilla FX Options market
- Correlation coefficients ← ????

- **Correlation is not quoted, not traded, not observable**
- **Solution: Use the dependence of FX spots**

Triangular Relationship

- **(1) GBP/USD**
- **(2) USD/JPY**
- **(3) GBP/JPY**

$$S_t^{(1)} \cdot S_t^{(2)} = S_t^{(3)}$$

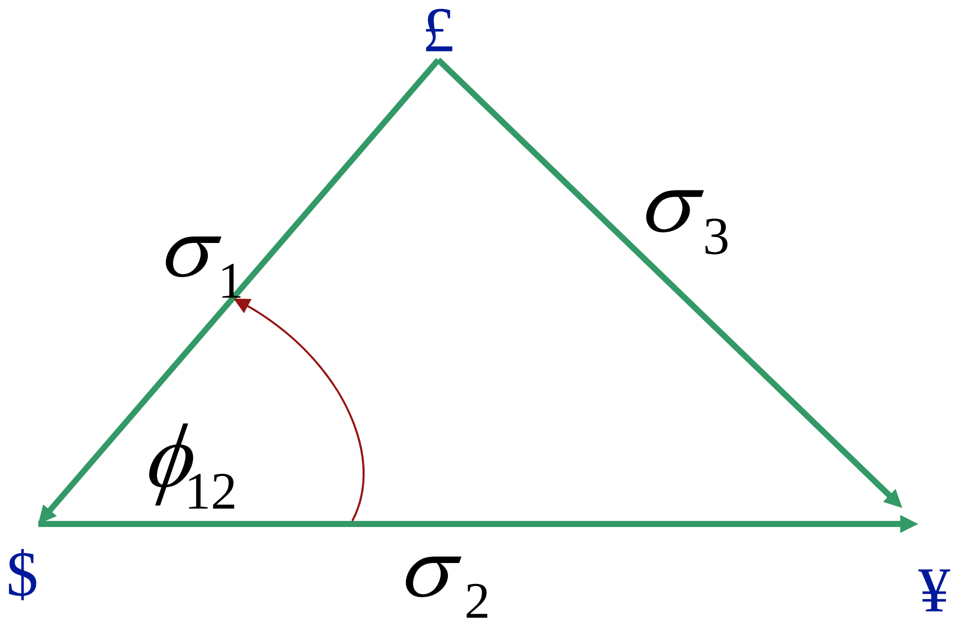
$$\Rightarrow \operatorname{var} \log S_t^{(1)} + \operatorname{var} \log S_t^{(2)} +$$

$$2 \operatorname{cov}(\log S_t^{(1)}, \log S_t^{(2)}) = \operatorname{var} \log S_t^{(3)}$$

$$\Rightarrow \sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12} = \sigma_3^2$$

$$\Rightarrow \rho_{12} = \frac{\sigma_3^2 - \sigma_1^2 - \sigma_2^2}{2\sigma_1\sigma_2}$$

Triangular Relationship: Geometric Interpretation



- **(1) GBP/USD**
- **(2) USD/JPY**
- **(3) GBP/JPY**

$$S_t^{(1)} \cdot S_t^{(2)} = S_t^{(3)}$$

$$-\rho_{12} = \cos\phi_{12}$$

Law of Cosine

$$\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\cos\phi_{12} = \sigma_3^2$$

Extensions

- **What is the correlation between USD/JPY and GBP/EUR?**
- **Does the law of cosine work in higher dimensions like in a tetrahedron?**
- **What else do we need?**
- **Does the method extend to equity options?**

FX Market in a Tetrahedron

- We need 15 correlation coefficients
- 12=3*4 from triangular markets
- The remaining 3 are

$$\rho_{16}, \rho_{25}, \rho_{34}$$

- (1) GBP/USD
- (2) USD/JPY
- (3) GBP/JPY
- (4) EUR/USD
- (5) EUR/GBP
- (6) EUR/JPY

FX Market in a Tetrahedron

Example ρ_{34}

$$S_t^{(4)} \cdot S_t^{(2)} = S_t^{(6)}$$

$$\Rightarrow \operatorname{cov}(\log S_t^{(3)}, \log S_t^{(4)}) = \operatorname{cov}(\log S_t^{(3)}, \log S_t^{(6)}) - \operatorname{cov}(\log S_t^{(3)}, \log S_t^{(2)})$$

$$\Rightarrow \sigma_3 \sigma_4 \rho_{34} = \sigma_3 \sigma_6 \rho_{36} - \sigma_3 \sigma_2 \rho_{23}$$

$$= \frac{1}{2}\left(\sigma_3^2 + \sigma_6^2 - \sigma_5^2\right) - \frac{1}{2}\left(\sigma_3^2 + \sigma_2^2 - \sigma_1^2\right)$$

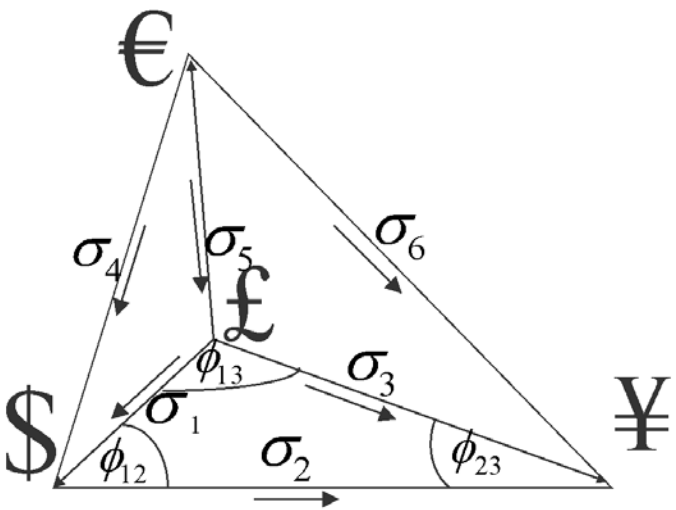
$$= \frac{1}{2}\left(\sigma_1^2 + \sigma_6^2 - \sigma_2^2 - \sigma_5^2\right)$$

$$\Rightarrow \rho_{34} = \frac{\sigma_1^2 + \sigma_6^2 - \sigma_2^2 - \sigma_5^2}{2\sigma_3\sigma_4}$$

- (1) GBP/USD
- (2) USD/JPY
- (3) GBP/JPY
- (4) EUR/USD
- (5) EUR/GBP
- (6) EUR/JPY

Correlation Risk of Multi-Currency Options

- Correlation coefficients in FX Markets can be inferred from cross instruments
- Correlation risk can be transferred into vega positions
- … and hence be hedged with vanilla options



$$\rho_{12} = \frac{\sigma_3^2 - \sigma_1^2 - \sigma_2^2}{2\sigma_1\sigma_2}$$

$$\rho_{34} = \frac{\sigma_1^2 + \sigma_6^2 - \sigma_2^2 - \sigma_5^2}{2\sigma_3\sigma_4}$$

Hedging Correlation Risk with Vanilla Options

- **Value of a rainbow option:** $R(\sigma, \rho)$
- **Now we know:** $\rho = \rho(\sigma)$
- **Write the value as:** $H(\sigma) = R(\sigma, \rho(\sigma))$
- **Plain Vega:** $\frac{\partial R}{\partial \sigma_i}$
- **Adjusted Vega:**

$$\frac{\partial H}{\partial \sigma_i} = \frac{\partial R}{\partial \sigma_i} + \sum_{j=1}^{5} \sum_{k=j+1}^{6} \frac{\partial R}{\partial \rho_{jk}} \frac{\partial \rho_{jk}}{\partial \sigma_i}$$

$$\frac{\partial \rho_{34}}{\partial \sigma_1} = \frac{\sigma_1}{\sigma_3 \sigma_4}$$

Extensions

- What is the correlation between USD/JPY and GBP/EUR? Done ! ☺
- Does the law of cosine work in higher dimensions like in a tetrahedron? No doubt! ☺
- What else do we need? Nothing ☺
- Does the method extend to equity options? Rather not ☹ Use correlation swaps for hedging

How to get the Smile into the Basket?

- Weighted Monte Carlo
- Works well for baskets up to 10 constituents
- Reference: Avellaneda, Buff, Friedman, Grandchamp, Kruk, Newman: Weighted Monte Carlo: A new technique for calibrating asset-pricing models
- Alternative: Optimal Strike Decomposition