

On the Cost of Poor Volatility Modeling

The Case of Cliquets

Fiodar Kilin* Morten Nalholm† Uwe Wystup‡

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Abstract

We have conducted pricing and hedging experiments in order to check whether simple stochastic volatility models are capable of capturing the forward volatility and forward skew risks correctly. As a reference we have used the Bergomi model that treats these risks accurately per definition. Results of our experiments show that the cost of poor volatility modeling in the Heston model, the Barndorff-Nielsen-Shephard model and a Variance-Gamma model with stochastic arrival is too high when pricing and hedging cliquet options.

1 Introduction

When pricing exotic options, as in any modeling effort, the art in model choice is to pick the simplest model that reasonably captures the behavior of the risks to which the products are

*Frankfurt School of Finance & Management, Quanteam AG, fiodar.kilin@quanteam.de

†Department of Mathematical Sciences, University of Copenhagen, nalholm@math.ku.dk

‡Frankfurt School of Finance & Management, MathFinance AG, uwe.wystup@mathfinance.com

sensitive. The impact of such model risk on pricing can be substantial, see e.g. Hirta, Courtadon & Madan (2002), for an illustration of the impact of model risk on hedging of exotics see Nalholm & Poulsen (2006).

The balance between the complexity of a model and its ability to appropriately capture the relevant phenomena is furthermore affected by the need for tractability. An example of a debate on how to strike this balance was initiated by Longstaff, Santa-Clara & Schwartz (2001) and Andersen & Andreasen (2001) in an interest rate setting. As indicated in these papers, the cost of a poor model choice may be substantial.

In the competition for costumers and market shares dealers are continuously engineering new products to attract interest and meet the needs of clients. Some innovative structured equity derivatives are highly forward-volatility and/or forward-skew dependent. Examples of such products are certain cliquet options.

When these structures were initially introduced their risks were apparently not fully understood. Reports in the industry press suggest that this lead some dealers to price the products well below the cost of the hedge (see e.g. Jeffrey (2004)).

The purpose of this paper is to investigate the cost of using popular models for pricing and hedging cliquet options when applied in a more complicated reality. Recently, a few attempts have been made to construct models that adequately capture the behavior of the risks that cliquet options are exposed to (see e.g. Bergomi (2005) and Buehler (2006)). To gain insights into the cost of using simpler models to price and hedge cliquet options, we assume that the true data-generating process is produced by the model in Bergomi (2005). We assume therefore, that derivatives values in this model are *true* prices. Then, we use a number of popular models to price and hedge cliquet options. This experimental design is in the spirit of Hull & Suo (2002). The performance of the popular models is measured by relating their theoretical prices to the

true price and by investigating their hedge error distributions. This experiment in a controlled environment allows us to concentrate on the cost incurred in case of too simple modeling of volatility dynamics.

The rest of the paper is structured as follows: Definitions of forward volatility and forward skew are introduced in section 2. Examples and properties of cliquet options are discussed in section 3. The models and the relevant calibration methods are described in section 4. We conduct a price comparison in section 5. The performance of hedge strategies based on simpler popular models is analyzed in section 6 and section 7 concludes.

2 Forward Volatility and Forward Skew

Analysis of influence of implied volatility dynamics on pricing and hedging exotic options should take into account sensitivity of a particular product to forward volatility and forward skew. However, we should be careful when using the terms forward volatility and forward skew. Sometimes these terms are used without defining them. It can be confusing since there are alternative non-equivalent definitions of forward volatility. We should distinguish between these definitions. We should also distinguish between *forward-vol-sensitive options* and *forward-skew-sensitive options*. We may give the following non-equivalent definitions of forward volatility (forward implied volatility):

Definition 1. *Forward volatility* is the implied volatility for a relative strike k and maturity T_2 that is observed at a future time-point T_1 . This value is unknown today. It is also model-independent: one needs no model to observe this value - one needs to wait only until time T_1 . We call this forward volatility the *future volatility*.

Definition 2. *Forward volatility* is the Black-Scholes volatility implied from a price of a forward-start call computed in another model. In other words: At time t compute the price C_F of

a forward-start call maturing at T_2 and relative strike k , which will be fixed at T_1 , in some model. Find the volatility $\sigma_f^{T_1, T_2}(t, k)$, such that the Black-Scholes price of this forward-start call with the volatility $\sigma_f^{T_1, T_2}(t, k)$ is equal to C_F . The value $\sigma_f^{T_1, T_2}(t, k)$ is *forward volatility*. This value is known today and it is model-dependent. We will refer to this forward volatility as the *forward-start-vanilla-implied forward volatility*.

Definition 3. *Forward volatility* is the forward variance swap volatility process

$$\sigma_f^{T_1, T_2}(t) = \sqrt{\xi^{T_1, T_2}(t)} = \sqrt{\frac{(T_2 - t)V_t^{T_2} - (T_1 - t)V_t^{T_1}}{T_2 - T_1}},$$

where $V_t^{T_1}$ and $V_t^{T_2}$ are the implied variance swap variances, $t < T_1 < T_2$. The starting value $\sigma_f^{T_1, T_2}(0)$ of this process is known today. This definition is convenient for modeling and is used in the Bergomi model described below. We will refer to this forward volatility as the *variance-swap-implied forward volatility*.

Although these definitions are not equivalent, all these definitions are suited for the definition of the term *forward-volatility-sensitive option* and reflect the same qualitative aspect of the implied volatility dynamics.

Definitions 1 and 2 can be adapted for the definition of *forward skew*. We can define intuitively:

$$\text{forward skew} = \frac{\sigma_f^{T_1, T_2}(t, k_2) - \sigma_f^{T_1, T_2}(t, k_1)}{k_2 - k_1},$$

or exactly:

$$\text{forward skew} = \left. \frac{d\sigma_f^{T_1, T_2}(t, k)}{d \ln k} \right|_{k = \frac{F(t, T_2)}{F(t, T_1)}},$$

where σ_f is the forward volatility and $F(t, \tau)$ is the forward price for maturity τ calculated at time point t . This reflects the slope of the smile curve as a function of the strike and is related to the risk reversal quotes in the market.

A simple example of a forward-volatility-sensitive option is a forward-start call. A simple example of a forward-skew-sensitive option is a forward-start call spread. Further examples of forward-volatility- and forward-skew-sensitive options are introduced in the next section.

3 Cliquet Options

A cliquet option is a derivative that pays off some function of a set of relative returns of an underlying. Typically this function incorporates local or global caps and floors, minimum or maximum functions, sums and fixed coupons. The relative returns are typically calculated on a monthly, semi-annual or annual basis.

Wilmott (2002) shows an example where the sensitivity of a cliquet option to a deterministic volatility is negligible in comparison with the real sensitivity of this option to volatility dynamics. To show this fact, he considers a globally floored, locally capped cliquet and analyzes a model where the actual volatility is chosen to vary in such a way as to give the option its highest or lowest possible value. This model exploits the property that an increase in volatility leads to an increase of the option value when Gamma is positive and to a decrease of the option value when Gamma is negative.

Schoutens, Simons & Tistaert (2004) compare prices of cliquet options in seven stochastic volatility models calibrated to the implied volatility surface of the Eurostoxx 50 index. They observe a price range of more than 40 percent amongst these models. They show that different models can produce almost the same marginal distribution of the underlying, but at the same time totally different cliquet prices. They demonstrate how the fine-grain structure of the underlying process influences the exotic option values.

Here we introduce definitions of particular cliquet options that are referred to in this paper.

3.1 Reverse Cliquet

The payoff of a reverse cliquet is

$$\max\left(0, C + \sum_{i=0}^{N-1} r_i^-\right), \quad (1)$$

where

$$\begin{aligned} r_i &= \frac{S_{T_{i+1}} - S_{T_i}}{S_{T_i}}, \\ r_i^- &= \min(r_i, 0), \\ 0 &= T_0 < T_1 < T_2 < \dots < T_N, \\ C &> 0. \end{aligned}$$

In this paper, all numerical examples for the reverse cliquet have been calculated using the same contract specification as in Bergomi (2005). The length of the reset period $T_{i+1} - T_i$ is one month. The number of reset periods is $N = 36$, therefore the maturity is three years. The maximum possible payoff is equal to the coupon $C = 50\%$.

This option is called “reverse cliquet” because the final payoff depends on negative returns only. This option is both forward-volatility- and forward-skew-sensitive.

3.2 Napoleon

This contract consists of several building blocks. The payoff of each building block, which is settled separately, is

$$\max\left(0, C + \min_{i=0, N-1} r_i\right), \quad (2)$$

where

$$\begin{aligned}
 r_i &= \frac{S_{T_{i+1}} - S_{T_i}}{S_{T_i}}, \\
 0 &\leq T_0 < T_1 < T_2 < \dots < T_N, \\
 C &> 0.
 \end{aligned}$$

In this paper we consider an example of a Napoleon option that consists of three building blocks. Each building block has $N = 12$ reset periods. The length of each reset period $T_{i+1} - T_i$ is one month. Therefore the maturity of this contract is three years and the possible payments occur at the end of each year. The maximum possible payment at the end of each year is equal to the coupon $C = 8\%$.

This contract type is analyzed in Bergomi (2004), Bergomi (2005) and Gatheral (2006). It is extremely forward-volatility-sensitive but almost forward-skew-insensitive.

3.3 Accumulator

The payoff of an accumulator is

$$\max \left(0, \sum_{i=0}^{N-1} \max(\min(r_i, \text{cap}), \text{floor}) \right), \tag{3}$$

where

$$\begin{aligned}
 r_i &= \frac{S_{T_{i+1}} - S_{T_i}}{S_{T_i}}, \\
 0 &= T_0 < T_1 < T_2 < \dots < T_N.
 \end{aligned}$$

In this paper we consider an example of an accumulator where the floor is set to -1% , the cap is 1% , each reset period $T_{i+1} - T_i$ is one month and the maturity of the option is three years ($N = 36$). This option is forward-skew-sensitive but almost forward-volatility-insensitive.

3.4 Call Spread Cliquet

This option pays at the end of each reset period $[T_i, T_{i+1}]$ the amount

$$\max\left(\frac{S_{T_{i+1}}}{S_{T_i}} - k_1, 0\right) - \max\left(\frac{S_{T_{i+1}}}{S_{T_i}} - k_2, 0\right). \quad (4)$$

A call spread cliquet can be seen as a portfolio of forward start call spreads. As in the previous examples we consider $N = 36$ monthly reset periods $T_{i+1} - T_i$. The strikes are set to $k_1 = 0.95$ and $k_2 = 1.05$. This option inherits forward-skew-sensitivity from its forward-start call-spread building blocks.

4 Models

It is well known that many exotic options, even simple (reverse) barrier options, are sensitive to the shape of future implied volatility surfaces. A discussion of this issue for cliquet options can be found in Bergomi (2004).

It is also well known that many of the popular option pricing models are regrettably not able to properly capture the observed dynamics of the implied volatility surfaces. This is true even though the models are reasonably good according to the traditional metric, namely the ability to calibrate to observed implied volatility surfaces.

Motivated by these observations an option pricing model where the dynamics of the variance swap variances is modelled directly was suggested by Bergomi (2005). Here we outline the definition of this model.

4.1 Bergomi Model

The dynamics of forward variances is modelled for discrete time intervals $[T_i, T_{i+1}]$, where $T_i = t_0 + i\Delta, i = 0, \dots, N$. The time step Δ is typically equal to a reset period of a cliquet option that we need to price and hedge. A set of the forward variance processes is defined as

$$\xi^{T_i}(t) = \frac{(T_{i+1} - t)V_t^{T_{i+1}} - (T_i - t)V_t^{T_i}}{\Delta}, \quad 0 \leq t \leq T_i, \quad (5)$$

where V_t^T is the implied variance swap variance observed at time t for maturity T . Initial values of the implied variance swap variances $\{V_{t_0}^{T_i}, i = 0, \dots, N\}$ are used as an input to the Bergomi model. Initial forward variances $\{\xi^{T_i}(t_0), i = 0, \dots, N\}$ are calculated from this input using equation (5).

The dynamics of each forward variance process $\{\xi^{T_i}(t), i = 0, \dots, N\}$ is modelled as

$$d\xi^{T_i}(t) = \omega \xi^{T_i}(t) (e^{-k_1(T_i-t)} dU_t + \theta e^{-k_2(T_i-t)} dW_t), \quad (6)$$

$$\text{Cov}[dU_t, dW_t] = \rho dt. \quad (7)$$

$\xi^{T_i}(t)$ is a random process in the time interval $[t_0, T_i]$. At time T_i the variance swap variance for the interval $[T_i, T_{i+1}]$ is known to be

$$V_{T_i}^{T_{i+1}} = \xi^{T_i}(T_i). \quad (8)$$

The solution of the SDE (6) is

$$\begin{aligned} \xi^T(t) = \xi^T(0) \exp(\omega [e^{-k_1(T-t)} X_t + \theta e^{-k_2(T-t)} Y_t] \\ - \frac{\omega^2}{2} [e^{-2k_1(T-t)} E[X_t^2] + \theta^2 e^{-2k_2(T-t)} E[Y_t^2] + 2\theta e^{-(k_1+k_2)(T-t)} E[X_t Y_t]]), \end{aligned} \quad (9)$$

where

$$dX_t = -k_1 X_t dt + dU_t, \quad (10)$$

and

$$dY_t = -k_2 Y_t dt + dW_t. \quad (11)$$

Over the interval $[T_i, T_{i+1}]$ the risk-neutral dynamics of the underlying is

$$\frac{dS}{S} = r_t dt + \alpha_i S^{\beta_i - 1} dZ_t, \quad (12)$$

where r is the risk-free interest rate and the dividend yield is assumed to be zero. The parameters α_i and β_i are recalibrated when t reaches T_i so that

$$\sqrt{\xi^i(T_i)} = \sigma_{IV}^{CEV}(S_{T_i}, T_{i+1} - T_i, F, \alpha_i, \beta_i), \quad (13)$$

$$\chi + \nu \sqrt{\xi^i(T_i)} = \sigma_{IV}^{CEV}(S_{T_i}, T_{i+1} - T_i, 1.05F, \alpha_i, \beta_i) - \sigma_{IV}^{CEV}(S_{T_i}, T_{i+1} - T_i, 0.95F, \alpha_i, \beta_i), \quad (14)$$

where $F = S_{T_i} e^{r(T_{i+1} - T_i)}$ and $\sigma_{IV}^{CEV}(S_{T_i}, \tau, K, \alpha_i, \beta_i)$ is the Black-Scholes implied volatility calculated from the price of a call option in the CEV model (12) calculated at the time T_i . Here K and τ are strike and time to maturity of this option. The idea behind this approach is to have the model parameters χ and ν set by the responsible trader, where χ resembles a general level of the skew, which is called a risk reversal in the market and ν is inverse proportional to the overall volatility level. This way the skew becomes stochastic in addition to the volatility.

The underlying process is correlated with both factors that drive the forward variance process

$$\text{Cov}[dZ_t, dU_t] = \rho_{SX} dt, \quad (15)$$

$$\text{Cov}[dZ_t, dW_t] = \rho_{SY} dt. \quad (16)$$

As noted in Bergomi (2005), the model is currently difficult to use because of a lack of suitable calibration instruments. This will change if and once options on forward ATM or variance swap volatilities become standard products, thus qualitatively expanding the set of calibration instruments.

We take the model to be the true data-generating process. This assumption allows us to gauge the differences between this recent model and more simple popular models. The parameter

values we use are reported in Table 1. Initial implied variance swap variances are generated in the Heston model with the parameters $\sigma_{0Heston}^2$, η_{Heston} and κ_{Heston} . The meaning of these parameters is described in Subsection 4.2 about the Heston model.

Scen	ω	θ	k_1	k_2	ρ	ν	χ	ρ_{SX}	ρ_{SY}	η_{Heston}	$\sigma_{0Heston}^2$	κ_{Heston}
1	1.0	0.1	5.0	0.4	0.2	-0.2	-0.06	-0.6	-0.3	0.04	0.04	1.0
2	1.0	0.5	5.0	0.4	0.2	0.2	-0.06	-0.6	-0.3	0.09	0.04	1.0
3	1.0	0.2	4.8	0.4	0.2	0.2	-0.06	-0.6	-0.3	0.09	0.04	1.0
4	1.4	0.3	6.0	0.25	0.0	0.1	-0.07	-0.7	-0.35	0.05	0.04	1.0
5	1.5	0.3	6.0	0.25	0.0	0.2	-0.0625	-0.7	-0.35	0.09	0.04	1.0

Table 1: Parameter values for five scenarios used for the Bergomi model.

The popular models we consider are representative of three different approaches to model volatility clustering, namely diffusion models with stochastic volatility, non-Gaussian Ornstein-Uhlenbeck-based models and Levy models with a stochastic time-change. Specifically, we consider the models by Heston (1993), Barndorff-Nielsen & Shephard (2001) and Carr, Geman, Madan & Yor (2003). Here we outline the risk-neutral dynamics in these models and describe the influence of each model parameter on the form and dynamics of the implied volatility surface.

4.2 Heston Model

The risk-neutral dynamics in the Heston model is

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t, \quad S_0 \geq 0, \quad (17)$$

where

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t d\tilde{W}_t, \quad \sigma_0 \geq 0, \quad (18)$$

$$\text{Cov}[dW_t, d\tilde{W}_t] = \rho dt. \quad (19)$$

The parameter κ denotes the mean-reversion speed. The reciprocal of this parameter $\tau = \frac{1}{\kappa}$ separates short and long maturities in the sense that asymptotic expressions for the ATM (in the sense of strike being at the forward price) implied volatility and skew are valid for $t \ll \tau$ (short-term asymptotics) and $t \gg \tau$ (long-term asymptotics). The long-run variance η has a major impact on the long-term implied volatility surface ¹ - the long-term ATM implied volatility is approximately proportional to $\sqrt{\eta}$, the long-term skew is approximately inverse proportional to $\sqrt{\eta}$. The volatility of variance θ creates convexity in the implied volatility curves for each maturity and controls the dynamics of short-term implied volatilities. The correlation parameter ρ also has two different objectives in this model: it creates the skew (the slope of the implied volatility curves) for each maturity and measures the correlation between the underlying and the ATM implied volatility. The initial value of the instantaneous volatility σ is σ_0 . The state variable σ is not an observable in theory. However, in practice, this variable can be observed in liquid option markets using extrapolation in the implied volatility surface to the zero maturity and ATM strike. Such an extrapolation is justified by the short-term asymptotics of the implied volatility surface generated by the Heston model.

4.3 Barndorff-Nielsen-Shephard Model (BNS)

This model introduces simultaneous up-jumps in the volatility and down-jumps in the underlying price. The risk-neutral dynamics of the log-spot is

$$d(\log S_t) = (r - \lambda k(-\rho) - \sigma_t^2/2)dt + \sigma_t dW_t + \rho dz(\lambda t), \rho < 0, \quad (20)$$

with the latent state following the process

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dz(\lambda t), \quad (21)$$

¹The asymptotic expressions for the ATM implied volatility and skew in the Heston model can be found in Gatheral (2006) and Bergomi (2004).

where

$$z(t) = \sum_{n=1}^{N_t} x_n. \quad (22)$$

N_t is a Poisson process with intensity a , x_n is an i.i.d. sequence, each x_n follows an exponential law with mean $1/b$. The cumulant function of $z(1)$ is

$$k(u) = \ln E(\exp(-uz(1))) = -au(b+u)^{-1}. \quad (23)$$

In contrast to the Heston model, the short-term skew in the Barndorff-Nielsen-Shephard model is not explained by the dependency between the underlying and the latent state processes. In this model the short-term skew is generated only by the possibility of a jump in the underlying. Therefore the short-term skew is controlled by a triplet of parameters $\{a, b, \rho\}$. Since the comovement parameter ρ is assumed to be negative, this model produces a negative short-term skew. The long-term skew in the Barndorff-Nielsen-Shephard model is generated by the superposition of two effects: jumps in the underlying and dependency between two state processes S_t and σ_t . Therefore the long-term skew is more sensitive to the comovement parameter ρ than the short-term skew. The Poisson intensity parameter a has both static and dynamic objectives in this model. Its main static objective is to control the level of the long term ATM-volatility. The reason for this fact is that the process σ_t^2 is stationary and has a marginal law that follows a Gamma distribution with mean a and variance a/b . The dynamic objective of the parameter a is to control the dynamics of short-term implied volatilities. An intuitive comparison between the dynamic objectives of the Poisson intensity a in the Barndorff-Nielsen-Shephard model and the volatility of variance θ in the Heston model can be seen here. However, it should be taken into account that σ_t in the Barndorff-Nielsen-Shephard model cannot be interpreted as “volatility” since the term $\rho dz(\lambda t)$ also affects the returns.

4.4 Variance Gamma with Stochastic Arrival (VGSA)

This model introduces stochastic volatility effects by making the time stochastic. The risk-neutral underlying process is modeled as

$$S_t = S_0 \frac{\exp(rt)}{E[\exp(X_{Y_t})|y_0]} \exp(X_{Y_t}), \quad (24)$$

where X_t is a Variance Gamma process defined by three parameters: drift θ and volatility σ of the Brownian motion and the variance ν of the subordinator. Y_t is a business time

$$Y_t = \int_0^t y_s ds, \quad (25)$$

where y_t follows the CIR stochastic clock

$$dy_t = \kappa(\eta - y_t)dt + \lambda y_t^{1/2} dW_t. \quad (26)$$

In this model the short-term asymptotics is controlled by the three Variance Gamma parameters only. The short-term skew is mainly defined by the drift parameter² θ . The subordinator variance ν creates convexity in the implied volatility curves for short maturities. The CIR stochastic clock prevents the long-term skew from flattening too quickly³. Therefore the lower the CIR-long-term-mean parameter η , the higher the similarity between the long- and the short-term skews. The reciprocal of the CIR-mean-reversion rate κ separates short and long maturities in the same sense as the corresponding parameter in the Heston model. The dynamics of short-term ATM implied volatilities is mainly controlled by the CIR-volatility λ .

²Of course, this is only true if ν is positive. If ν is zero the Variance Gamma process becomes a Brownian motion, i.e. produces no skew.

³The main disadvantage of Levy models without stochastic arrival is that the implied volatility skew flattens too quickly. See e.g. Chapter 13 in Cont & Tankov (2004).

5 Price Comparison

In this section we compare the theoretical prices of cliquet options in the Bergomi model and in calibrated versions of the discussed more simple models. We use parameters reported in Table 1 for the Bergomi model to generate five implied volatility surfaces. Then we calibrate the more simple models to these implied volatility surfaces and compare price differences.

The calibrated parameters for the more simple models are reported in Tables 2-4. We have used the calibration algorithm described in Kilin (2006).

Of course, the calibrated values of mean-reversion rate, long-run variance and short-term volatility in the Heston model⁴ should not coincide with the corresponding parameters that we have used to generate initial implied variance swap variances. We cannot expect such a correspondence, because we use the set of vanilla options as a calibration input. In the absence of jumps⁵ a set of prices of vanilla options with all possible maturities and strikes contains more information than a set of prices of variance swaps with all possible maturities.

Calibrated values of the mean-reversion rate in the Heston model for scenarios 2 and 3 are exactly equal to 1.0. That is explained by the parameter bounds that have been used to calibrate the Heston model. For the mean-reversion rate we have used the bounds $[1.0; 4.0]$ which correspond to the time interval between three months and one year.

⁴See Table 2.

⁵Both Heston and Bergomi processes do not have jumps.

Scenario	η	κ	θ	σ_0	ρ
1	0.308888	1.524	2.57848	0.251838	-0.659371
2	0.0901768	1.0	0.231635	0.194986	-0.514877
3	0.0909267	1.0	0.149329	0.193474	-0.609845
4	0.053176	3.53349	0.521236	0.201211	-0.67917
5	0.091009	1.12328	0.277384	0.194058	-0.605952

Table 2: Calibrated Heston parameters.

Scenario	ρ	λ	b	a	σ_0
1	-5.17388	0.0403766	11.3553	9.98816	0.164977
2	-2.04324	0.83257	49.3421	4.00691	0.174382
3	-2.86624	0.804772	93.3514	7.78661	0.169629
4	-2.7184	2.0589	47.182	1.53604	0.159252
5	-1.99725	0.905518	36.6755	2.85488	0.168391

Table 3: Calibrated BNS parameters.

Scenario	θ	σ	ν	κ	η	λ	y_0
1	-0.19297	0.256868	0.273051	1.47153	1.56324	3.98489	1.0
2	-0.526388	0.19289	0.0210463	1.62598	0.941867	0.726751	1.0
3	-0.76172	0.184905	0.0153841	1.58304	0.912081	0.500934	1.0
4	-0.348248	0.197137	0.0573394	3.27292	0.712477	1.76894	1.0
5	-0.413335	0.193394	0.033752	1.7588	0.964997	1.01945	1.0

Table 4: Calibrated VGSA parameters.

Values of cliquet options in all described models are reported in Tables 5-9. These values have been calculated using a Monte-Carlo simulation with one million paths using antithetics as a variance reduction technique and a path generation of the variance process based on Andersen

& Brotherton-Ratcliffe (2005) in order to prevent the variance process to take negative values⁶. Using these values we generate 20 figures (5 scenarios x 4 instruments) that allow us to compare between different models. In 16 of these 20 figures the best approximation of the reference (Bergomi) price is the value calculated in the Heston model. However, in more than half of the pricing experiments the relative difference between reference price and value calculated in the Heston model is more than 3.5%, which is unacceptable for practical applications.

	Bergomi	Heston	BNS	VGSA
Napoleon	0.0595	0.0458	0.0331	0.0727
Reverse Cliquet	0.0258	0.0401	0.0203	0.0880
Accumulator	0.0666	0.0796	0.0424	0.1821
Call Spread Cliquet	2.0191	1.9389	1.9277	2.0165

Table 5: Comparison of theoretical cliquet values. Scenario 1.

	Bergomi	Heston	BNS	VGSA
Napoleon	0.0153	0.0148	0.0176	0.0274
Reverse Cliquet	0.0031	0.0025	0.0048	0.0144
Accumulator	0.0228	0.0231	0.0252	0.0446
Call Spread Cliquet	1.7776	1.7929	1.8087	1.8716

Table 6: Comparison of theoretical cliquet values. Scenario 2.

⁶In our experiments the highest standard error for reverse cliquets was 0.00002, for Napoleons 0.00004, for accumulators 0.00004, for call spread cliquets 0.0002.

	Bergomi	Heston	BNS	VGSA
Napoleon	0.0117	0.0113	0.0132	0.0254
Reverse Cliquet	0.0011	0.0016	0.0020	0.0115
Accumulator	0.0217	0.0216	0.0239	0.0410
Call Spread Cliquet	1.7763	1.7830	1.8004	1.8619

Table 7: Comparison of theoretical cliquet values. Scenario 3.

	Bergomi	Heston	BNS	VGSA
Napoleon	0.0282	0.0279	0.0374	0.0433
Reverse Cliquet	0.0137	0.0110	0.0245	0.0422
Accumulator	0.0422	0.0382	0.0480	0.0887
Call Spread Cliquet	1.8994	1.8658	1.9050	1.9323

Table 8: Comparison of theoretical cliquet values. Scenario 4.

	Bergomi	Heston	BNS	VGSA
Napoleon	0.0170	0.0163	0.0216	0.0305
Reverse Cliquet	0.0042	0.0034	0.0084	0.0199
Accumulator	0.0241	0.0254	0.0286	0.0555
Call Spread Cliquet	1.7829	1.8068	1.8294	1.9003

Table 9: Comparison of theoretical cliquet values. Scenario 5.

6 Hedge Performance

Practical implementations of the hedge strategies corresponding to the theoretical values analyzed in section 5 necessarily involve a number of approximations. The two main sources of discrepancies between theoretical and realized hedge performance are discrete rehedging and the practice of recalibrating the model used to compute the hedge ratios. In this section we analyze the magnitude of the discrepancies caused by these two sources.

We concentrate on the hedge performance of the Heston model only. There are two reasons for this choice. First of all, the results of the previous section have shown that the differences between cliquet values in the Heston model and in the reference (Bergomi) model are less than the corresponding differences for other models. Secondly, our analysis of the hedge performance requires a huge number of recalculations of cliquet values. It would be extremely time-consuming, if each value recalculation was done using the Monte-Carlo method. Fortunately, there is a much faster method to calculate the values of call spread cliquets in the Heston model. Lucic (2004) shows how to calculate the value of a forward-start option in the Heston model in a few milliseconds. Since a call spread cliquet can be seen as a portfolio of forward start call spreads, the same pricing method can be applied to calculate values of call spread cliquets.

The most natural way to evaluate the performance of a hedge strategy is to consider the realized profit-and-loss distribution. We construct such distributions by simulating the following implementations of the hedge strategies prescribed by the Heston model:

1. *Hedging with constant parameters (HCP)*: The Heston model is calibrated to the initial implied volatility surface and the subsequent hedge adjustments are done holding the calibrated parameters constant through time. This is in line with the assumptions underlying the Heston model.
2. *Hedging with recalibration (HR)*: The Heston model is recalibrated after each increment in

the Bergomi model. This is in line with standard practice.

To construct such distributions, we conduct simulation experiments where each iteration has the following structure:

1. Simulate an increment in a realization of the Bergomi model.
2. If relevant, calibrate the more simple models to the resulting implied volatility surface.
3. Adjust hedges according to the, possibly recalibrated, Heston model. The hedges are adjusted on a weekly basis.
4. At expiry, record hedge errors.

We investigate the performance of two dynamic hedging strategies

1. Delta and short-term vega (DSV),
2. Delta and parallel shift vega (DPV),

where short-term vega is the sensitivity of the option price to the parameter σ_0 of the Heston model, parallel shift vega is the sensitivity of the option price to a simultaneous shift of the parameters σ_0 and η so that

$$\Delta\sigma_0 = \sqrt{\eta + \Delta\eta} - \sqrt{\eta}. \quad (27)$$

The comparison between HCP and HR implementations is done on the basis of the DSV strategy. The DPV strategy is analyzed using the HCP implementation. Histograms of absolute cumulative hedging errors in the resulting three experiments (HR-DSV, HCP-DSV, HCP-DPV) are shown in Figures 1-3. The total number of hedge scenarios is 100 in the experiment HR-DSV, 299 in HCP-DSV, 2782 in HCP-DPV. The underlying path has been generated using parameters of

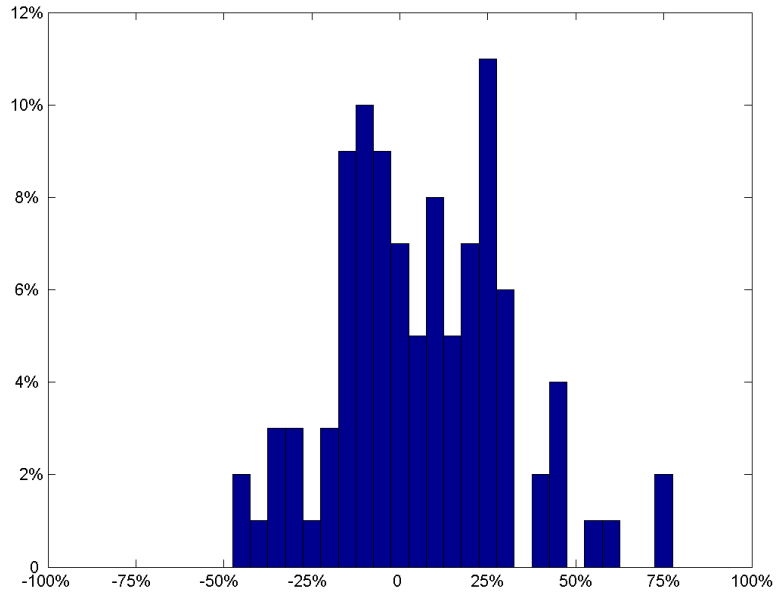


Figure 1: *Experiment HR-DSV. Histogram of relative cumulative hedging errors of a call spread cliquet.*

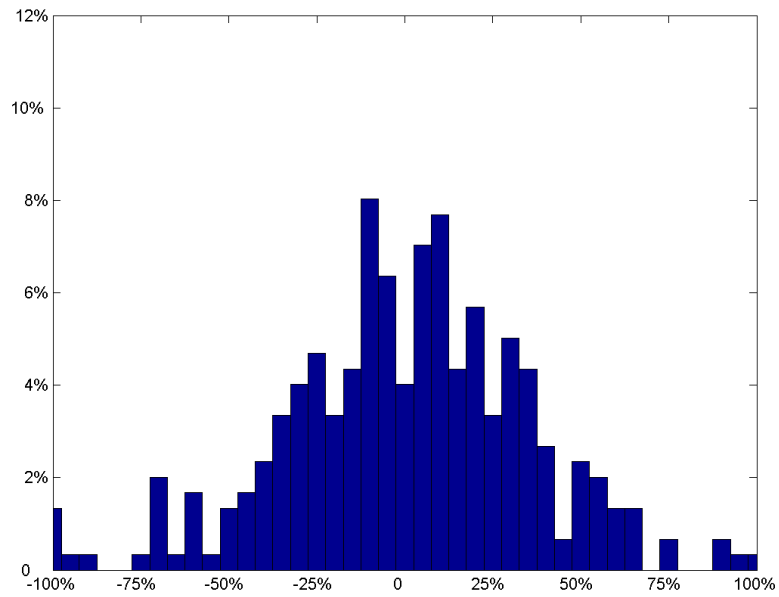


Figure 2: *Experiment HCP-DSV. Histogram of relative cumulative hedging errors of a call spread cliquet.*

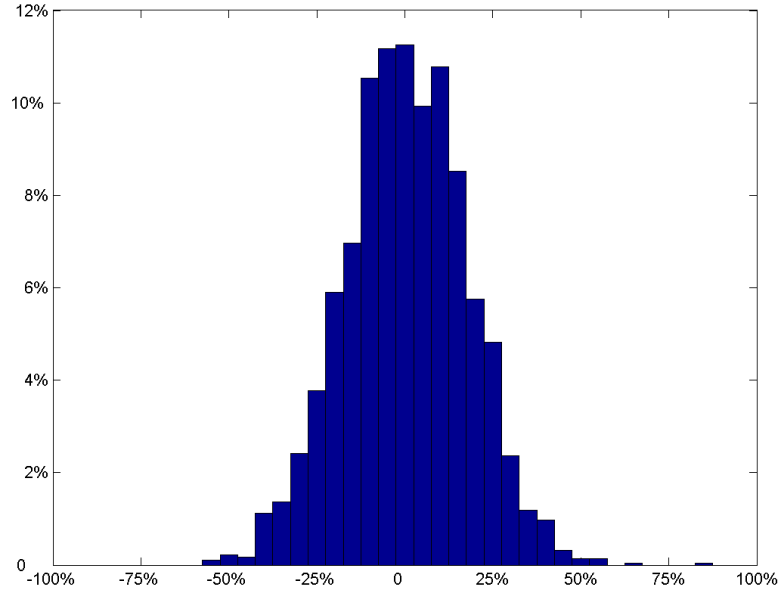


Figure 3: *Experiment HCP-DPV. Histogram of relative cumulative hedging errors of a call spread cliquet.*

scenario 5 (see Table 1). The option that has been hedged is a 36-periods 95-105% call spread cliquet with monthly resets. The value of the call spread cliquet in the Heston model at the issue date was 1.8068 in all experiments. Relative hedging errors are shown in percent of the cliquet value. Relative frequencies are shown in percent of the total number of experiments.

All experiments show that the hedging error can be unacceptably high. This observation confirms the assertion that the risk of using Heston's model for hedging cliquet options is too high.

7 Conclusion

The cost of poor volatility modeling in popular stochastic volatility models (Heston, Barndorff-Nielsen-Shephard, Levy with stochastic clock) is too high. A possible cause of relatively high cliquet hedging errors produced by these models is that they are developed to fit vanilla prices and

to control forward volatility and forward skew simultaneously. A lot of modeling and numerical effort is spent to reach all these targets in one model. However, to our knowledge, today there does not exist a model that attains both these targets simultaneously. Therefore, we would recommend to change the requirements for the model that should be used for pricing and hedging cliquet options. Specifically, we recommend to abandon the requirement to fit vanilla prices. This would facilitate direct modeling of forward volatility and forward skew. When pricing and hedging cliquets, accurate modeling of forward volatility, forward skew and vega-hedging are incomparably more important than fitting vanilla prices. Therefore we expect better hedging performance if we do not complicate models of smile dynamics by the requirement to fit vanilla prices. Bergomi's model is a good example of this approach. In our further research we plan to investigate whether it is possible to obtain acceptable cliquet hedging performance using direct forward smile modeling without fitting vanilla prices.

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